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No. I.

ON

CUBIC TRANSFORMATIONS.

BY

JOHN CASEY, LL.D., F.R.S., V.P.R.I.A.,

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"CUNNINGHAM MEMOIRS."

On Cubic Transformations. *By* John Casey, LL.D., F.R.S., V.P.R.I.A., Professor of Higher Mathematics and Mathematical Physics in the Catholic University of Ireland.

[Read, December 8th, 1879.]

CHAPTER I.

Section I.—Notation.

1. The following notation shall be used in this Memoir :—

1°. A point shall be denoted by a single letter, and its co-ordinates by the same letter with suffixes: thus the point x will be the point whose co-ordinates are x_1, x_2, x_3.

2°. The general equation of the n^{th} degree will be denoted by $(a_1x_1 + a_2x_2 + a_3x_3)^n$, or a_x^n, where, after the involution, a_1^n is replaced by the coefficient of x_1^n in the given function, $na_1^{n-1}a_2$ by the coefficient of $x_1^{n-1}x_2$, &c., &c. (See Clebsch's *Vorlesungen ueber Geometrie.*)

3°. The foregoing notation shall not be rigidly adhered to, for we shall frequently employ Cayley's quantic notation. The equation of the cubic which we shall work with is the canonical form—

$$x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0,$$

and this we shall call the fundamental cubic.

4°. If a conic A be circumscribed about a triangle, self-conjugate with respect to a conic B, we shall denote this relation by saying that A is conjugate to B (in German *vereinigt*).

2. The title of this Memoir, "Cubic Transformation," though the most appropriate I could find, is, I think, liable to some objection. *In the first place*, the Paper contains some investigations that could not well be included under its name; and, *secondly*, we do not in general, as might be expected from the analogy of conic sections, transform one cubic into another; but its methods were suggested by investigations on cubics, and we invariably make use of cubics or functions connected with them in our transformations. All this will fully appear in the course of the Memoir, to which we now proceed.

Section II.—The Hessian.

3. If $U = 0$, $V = 0$ be the equations of two curves, then the Hessian of $U + kV$ is

$$\begin{vmatrix} u_{11} + kv_{11}, & u_{12} + kv_{12}, & u_{13} + kv_{13}, \\ u_{21} + kv_{21}, & u_{22} + kv_{22}, & u_{23} + kv_{23}, \\ u_{31} + kv_{31}, & u_{32} + kv_{32}, & u_{33} + kv_{33}, \end{vmatrix} = 0, \qquad (1)$$

where u_{11}, u_{12}, &c., denote second differentials. The coefficient of k in this equation is the function

$$v_{11}(u_{22}.u_{33} - u_{23}^2) + v_{22}(u_{33}.u_{11} - u_{31}^2) + v_{33}(u_{11}.u_{22} - u_{12}^2) + 2v_{23}(u_{12}.u_{13} - u_{11}.u_{23})$$
$$+ 2v_{31}(u_{23}.u_{21} - u_{22}.u_{31}) + 2v_{12}(u_{31}.u_{32} - u_{33}.u_{12}). \qquad (2)$$

4. The expression (2) equated to zero evidently denotes the locus of points whose polar conics with respect to V are conjugate to the polar conics of U. Similarly, the coefficient of k^2 in the expansion of the determinant (1) is the locus of points whose polar conics with respect to U are conjugate to the polar conics of V. When U and V are cubics, these will be cubics; but if U be of the m^{th} and V of the n^{th} degree, these curves will be respectively of the degree

$$2m + n - 6, \qquad m + 2n - 6.$$

5. If, in Art. 3, V denote the cube of the line $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, or, say the line λ_r, then $U + kV$ denotes a curve osculating U in all

the points where it meets λ_x, that is in m points, and the Hessian of this osculating curve is

$$H + kE\lambda_x. \tag{3}$$

Where H is the Hessian of U, and E is the locus of points whose polar conics touch the line λ_x; then E will be of the degree $2(m-2)$, and this will be a conic when U is a cubic.

6. If V be the conic $(a, b, c, f, g, h)(x_1, x_2, x_3)^2$, then equation (2) gives us the locus of points to whose polar conics with respect to U the conic V is conjugate—viz., this is

$$\begin{aligned} a(u_{22}\,.\,u_{33} - u_{23}^2) + b(u_{33}\,.\,u_{11} - u_{31}^2) + c(u_{11}\,.\,u_{22} - u_{12}^2) + 2f(u_{12}\,.\,u_{13} - u_{11}\,.\,u_{23}) \\ + 2g(u_{23}\,.\,u_{21} - u_{22}\,.\,u_{31}) + 2h(u_{31}\,.\,u_{32} - u_{33}\,.\,u_{12}) = 0. \end{aligned} \tag{4}$$

If V be the product of two lines λ_x and μ_x, then equation (4) is the expansion of the determinant

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \lambda_1, \\ u_{21}, & u_{22}, & u_{23}, & \lambda_2, \\ u_{31}, & u_{32}, & u_{33}, & \lambda_3, \\ \mu_1, & \mu_2, & \mu_3, & \end{vmatrix} = 0. \tag{5}$$

This denotes the locus of points to whose polar conics the lines λ_x and μ_x are conjugate. Finally, if the lines λ_x and μ_x coincide, we get the locus of points whose polar conics touch the line λ_x. This is

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \lambda_1, \\ u_{21}, & u_{22}, & u_{23}, & \lambda_2, \\ u_{31}, & u_{32}, & u_{33}, & \lambda_3, \\ \lambda_1, & \lambda_2, & \lambda_3, & \end{vmatrix} = 0. \tag{6}$$

7. If the curves U and V, in Art. 3, be the fundamental cubic and its Hessian, we find the coefficient of k in equation (1), after an easy reduction, to be

$$-2SU, \tag{7}$$

where S is the quartic invariant of the cubic. Hence we have the following theorem:—*The polar conic of any point on a cubic, with respect to its Hessian, is conjugate to the polar conic of the same point, with respect to the cubic itself.*

On the same hypothesis, the coefficient of k^2 is found to be

$$-(TU - 12SH), \tag{8}$$

where T is the sextic invariant of the cubic. *Hence the locus of points, whose polar conics with respect to the cubic are conjugate to its polar conics with respect to the Hessian, is the curve* $TU - 12SH = 0$.

8. The equation (6) written in full contains both point and line co-ordinates. German mathematicians call such expressions "*zwischenformen*," the English "*mixed concomitants*." We shall have several such forms in the course of this Memoir, and we shall call them by the German name. The curve (6) is the one denoted by E in equation (3). It is denoted in Cayley's symbolical notation by $\overline{\lambda 12^2}$, and in Clebsch's by $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$: see Salmon's *Algebra*, page 17. The equation (5) is denoted by the same authors by $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$. These notations are very compact. We shall, in the next chapter, get very expressive names to denote these curves. Thus $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$ will be the Hessian transformation of λ_x, and $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ will be the Hessian transformation of the product of λ_x and μ_x.

9. If the line λ_x pass through a given point, say, the intersection of the lines λ'_x and λ''_x, we have $\lambda_i = \lambda'_i + k\lambda''_i$. Hence, substituting, we get an equation of the form

$$L + 2kR + k^2M = 0, \tag{9}$$

where

$$L = \begin{pmatrix}\lambda'\\ \lambda'\end{pmatrix}, \quad R = \begin{pmatrix}\lambda'\\ \lambda''\end{pmatrix}, \quad M = \begin{pmatrix}\lambda''\\ \lambda''\end{pmatrix}.$$

10. The discriminant of the equation (9) is

$$LM - R^2 = 0, \tag{10}$$

or

$$\binom{\lambda'}{\lambda'} \times \binom{\lambda''}{\lambda''} - \binom{\lambda'}{\lambda''}^2,$$

and this last is equal to

$$H\begin{pmatrix} \lambda' & \lambda'' \\ \lambda' & \lambda'' \end{pmatrix}, \tag{11}$$

where H is the Hessian of the fundamental curve, and

$$\begin{pmatrix} \lambda' & \lambda'' \\ \lambda' & \lambda'' \end{pmatrix}$$

denotes the determinant

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \lambda'_1, & \lambda''_1, \\ u_{21}, & u_{22}, & u_{23}, & \lambda'_2, & \lambda''_2, \\ u_{31}, & u_{32}, & u_{33}, & \lambda'_3, & \lambda''_3, \\ \lambda'_1, & \lambda'_2, & \lambda'_3, & & \\ \lambda''_1, & \lambda''_2, & \lambda''_3, & & \end{vmatrix}. \tag{12}$$

The determinant (12) is evidently the second polar of the point of intersection of $\lambda' x$, $\lambda'' x$. Hence, denoting the polar by P_2, we see that the envelope of the variable curve

$$L + 2kR + k^2M$$

is the curve

$$P_2H = 0. \tag{13}$$

11. From the form of equation (10) we see that each of the curves L and M touches H in every point where they meet it. Now, if the fundamental curve U is of the n^{th} degree, the curves L and M are each of the degree $2(n-2)$, and the Hessian of the degree $3(n-2)$. Hence L and M touch the Hessian in $3(n-2)^2$, and the curve R passes through each of these points.

We shall give an independent proof of the theorems of this Article in the next chapter.

12. Since the Hessian is generated as the envelope of the curve $L + 2kR + k^2M$, which has but one variable parameter (k), and since this generating curve touches the envelope in $3(n-2)^2$ points, we may for shortness call this system of points the (k) points. Again, the equation $LM - R^2 = 0$ is the locus of points which satisfy simultaneously the system of determinants

$$\begin{vmatrix} L, & R, & M \\ k^2, & k, & 1 \end{vmatrix}.$$

Hence the result of substituting the co-ordinates of any of the points of contact in the curves L, R, M is proportional to k^2, k, 1. Hence, if $L + 2kR + k^2M$, $L + 2k'R + k'^2M$ be two generating curves, the equation of a curve passing through their $6(n-2)^2$ points of contact will be given by the determinant

$$\begin{vmatrix} L, & R, & M, \\ k_2, & k, & 1, \\ k'^2, & k', & 1, \end{vmatrix} = 0;$$

or expanded,

$$L - (k + k')R + kk'M = 0. \tag{14}$$

Cor.—The curve $L - (k + k')R + kk'M$ also passes through the points of contact of the generating curves $L - 2kR + k^2M$, $L - 2k'R + k'^2M$ with P_2 (see Art. 10); that is, through $2(n-2)^2$ points, or $(n-2)^2$ points for each curve.

13. The curve $LM - R^2$ can be discussed exactly in the same way as I have done in my *Bicircular Quartics*, pp. 96–104, which is, in fact, but an extension of the method given in Salmon's *Conic Sections*, p. 248. We give here a couple of examples:—

1. Find the envelope of the curve which passes through the points of contact of the curves

$$L - (2k \tan\phi)R + (k^2 \tan^2\phi)M,$$
$$L - (2k \cot\phi)R + (k^2 \cot^2\phi)M,$$

with their envelope when ϕ is a constant angle but k variable.

From equation 14 we find the required curve to be the envelope of the curve

$$L - (2k \operatorname{cosec} 2\phi)R + k^2 M,$$

and hence it is the curve

$$LM - R^2 \operatorname{cosec}^2 2\phi = 0. \qquad (15)$$

2. Find the locus of the points of intersection of the same curves.

Eliminating R we get

$$k^2 = \frac{L}{M};$$

$$\therefore\ L \sec^2\phi = 2\sqrt{\frac{L}{M}} \cdot \tan\phi \,.\, R;$$

or

$$LM = R^2 \sin^2 2\phi. \qquad (16)$$

14. The points where the curve $\binom{\lambda}{\lambda}$, or $\overline{\lambda 12^2}$, touches the Hessian of the fundamental curve are the points which correspond to the $3(n-2)^2$ points when the line λ_x meets the *Steinerian* of the given curve.

Demonstration.—Let J be one of the points of contact of $\overline{\lambda 12^2}$ with the Hessian, then the polar conic of J has a double point which lies on the *Steinerian*; but the same polar conic (see Art. 6) touches λ_x: hence λ_x meets the *Steinerian* in the point which corresponds to J: hence the proposition is proved.

15. The polar conics of any point with respect to two cubics, U and $U + L^3$, where $L = \lambda_x$, have double contact, the chord of contact being the line $\lambda_x = 0$.

Demonstration.—Let the point be (α), then performing the operation

$$\alpha_1 \frac{d}{dx_1} + \alpha_2 \frac{d}{dx_2} + \alpha_3 \frac{d}{dx_3}$$

on the two cubics, the proposition is evident.

Cor.—If U be a curve of the n^{th} degree, we may prove in like manner that the m^{th} polar of any point with respect to U and $U + L^n$ will be two curves which have $(n - m)$ points of $(n - m)$ pointic contact, all the contacts lying on the line $L = 0$.

16. *Two cubics, $U - L^3$ and $U - M^3$ being given, both inscribed in the same cubic U, it is required to find the locus of a point whose polar conics with respect to these cubics touch each other.*

Performing the operation

$$\alpha_1 \frac{d}{dx_1} + \alpha_2 \frac{d}{dx_2} + \alpha_3 \frac{d}{dx_3}$$

on $U - L^3$, we get, after rejecting the factor 3, the polar conic of (α) to be

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + 2m(\alpha_1 x_2 x_3 + \alpha_2 x_3 x_1 + \alpha_3 x_1 x_2) - \alpha_1\lambda_1 + (\alpha_2\lambda_2 + \alpha_3\lambda_3)L^2 = 0.$$

Now, putting for shortness

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 = c^2,$$

our equation may be written

$$S - (cL)^2 = 0;$$

or, incorporating the c,

$$S - L^2 = 0;$$

but this is the result of clearing

$$S^{\frac{1}{2}} - L$$

of radicals. Similarly we get

$$S^{\frac{1}{2}} - M:$$

hence, multiplying the latter by k, and adding, we get

$$(1 + k)S^{\frac{1}{2}} - (L + kM) = 0;$$

and, cleared of radicals, we get

$$(\alpha_1,\ \alpha_2,\ \alpha_3,\ m\alpha_1,\ m\alpha_2,\ m\alpha_3)\ (x_1,\ x_2,\ x_3)^2 - (l_1x_1 + l_2x_2 + l_3x_3)^2,$$

where

$$l_1 = \frac{c\lambda_1 + kc_1\mu_1}{1+k},\ \&\text{c}.$$

Hence, forming the discriminant, and putting $u_{11},\ u_{22},\ u_{33},\ u_{23},\ u_{31},\ u_{12}$ for $\alpha_1,\ \alpha_2,\ \alpha_3,\ m\alpha_1,\ m\alpha_2,\ m\alpha_3$, we find, after clearing of fractions,

$$(1+k)^2H - \begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & c\lambda_1 + kc_1\mu_1, \\ u_{21}, & u_{22}, & u_{23}, & c\lambda_2 + kc_1\mu_2, \\ u_{31}, & u_{32}, & u_{33}, & c\lambda_3 + kc_1\mu_3, \\ c\lambda_1 + kc_1\mu_1, & c\lambda_2 + kc_1\mu_2, & c\lambda_3 + kc_1\mu_3 . & 0, \end{vmatrix}$$

or

$$(H - c^2\Sigma) + 2k(H - cc_1\Pi) + k^2(H - c_1{}^2\Sigma^1); \tag{17}$$

where

$$\Sigma = \binom{\lambda}{\lambda},\quad \Pi = \binom{\lambda}{\mu},\quad \Sigma^1 = \binom{\mu}{\mu}$$

(see notation, Art. 9); and the discriminant, with respect to k, is

$$(H - c^2\Sigma)\,(H - c_1{}^2\Sigma^1) - (H - cc_1\Pi)^2 = 0; \tag{18}$$

and this is the condition of contact at the polar conics of the point (α).

17. The last equation can be reduced as follows: after multiplying and reducing by the condition $\Sigma\Sigma^1 - \Pi^2 = HP$ (see Art. 10), where P is the polar line of the point (α), and rejecting the factor H, we get

$$c^2\Sigma + c_1{}^2\Sigma^1 - 2cc_1\Pi - c^2c_1{}^2P = 0.$$

Now,

$$c^2 = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3,$$
$$c_1{}^2 = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3.$$

Hence, restoring these values, and putting $x_1,\ x_2,\ x_3$ for $\alpha_1,\ \alpha_2,\ \alpha_3$, we see that we may put L and M for c^2, $c_1{}^2$, and we get, after clearing of radicals,

$$(L\Sigma + M\Sigma' - LMP)^2 - 4LM\Pi^2 = 0).$$

This equation may be written

$$(L\Sigma - M\Sigma')^2 - 2LMP(L\Sigma + M\Sigma') + (LMP)^2 + 4LM(\Sigma\Sigma' - \Pi_2) = 0\,;$$

or (see Art. 10),

$$(L\Sigma' - M\Sigma')^2 + 2LMP(2H - L\Sigma - M\Sigma') + (LMP)^2 = 0.$$

Now, the Hessian of $U - L^3$ is $H - L\Sigma$,

" $U - M^3$ is $H - M\Sigma'$;

hence, denoting these Hessians by H_1, H_2, we finally get a result equivalent to the polyzomal curve,

$$\sqrt{H_1} + \sqrt{H_2} + \sqrt{LMP} = 0, \tag{19}$$

cleared of radicals. Hence, this is the required locus, and being of the sixth degree, we see from its form that it has nine contacts with each of the curves H_1 and H_2, and has each of the lines L, M, P as a triple tangent.

18. From equation (17) we see the condition that the polar conics will cut harmonically is

$$H - cc_1\Pi = 0\,;$$

and, clearing of radicals, this is

$$H^2 - LM\Pi^2 = 0.$$

Hence, eliminating Π^2 from this, and the equation

$$\Sigma\Sigma' - \Pi^2 = PH,$$

we get, after an easy reduction,

$$H(H_1 + H_2 + LMP) - H_1H_2 = 0\,; \tag{20}$$

and this is the required locus. Hence we have the following theorem :—

The polar conics of any point (α) *in the curve*

$$H(H_1 + H_2 + LMP) - H_1H_2 = 0$$

with respect to the three cubics U, $U - L^3$, $U - M^3$, *are three conics of the*

form S, $S - L^2$, $S - M^2$, *and the pencil is harmonic, which is formed of the lines* L, M, *and the chords of contact of the two pairs of lines which can be drawn through the intersection of the line* $L - M$ *with* $S - L^2$ *to touch* S. (See *Bicircular Quartics*, Art. 128.)

19. *To find the relation, which must be satisfied, in order that the polar conic of a point with respect to the curve,* $U - L^3$, *may touch the polar conics of the same point with respect to the curves* $U - M'^3$, $U_1 - M''^3$, $U - M'''^3$.

From the equation (20) we have the co-existence of the three following equations as the required condition :—

$$(H - H_1) H_2 + HH_1 + HLM'P',$$
$$(H - H_1) H_2' + HH_1 + HLM''P'',$$
$$(H - H_1) H_2'' + HH_1 + HLM'''P'''.$$

Hence, eliminating, we get the determinant

$$\begin{vmatrix} H_2, & M'P', & 1, \\ H_2', & M''P'', & 1, \\ H_2'', & M'''P''', & 1; \end{vmatrix} = 0.$$

Now, $H_2 = H - M\Sigma'$, &c., and we get, finally, the following determinant—

$$\begin{vmatrix} \frac{1}{M'}, & \frac{1}{M''}, & \frac{1}{M'''}, \\ P', & P'', & P''', \\ \Sigma', & \Sigma'', & \Sigma''', \end{vmatrix} = 0. \tag{21}$$

SECTION III.—THREE CONJUGATE POINTS.

20. *Fundamental Theorem.—If* x, y, z *be three points, and* U *a cubic, then, if the polar line of* (y) *with respect to the polar conic of* (x) *with respect to the cubic* U *pass through* (z), *then the polar line of* (x) *with respect to the polar conic of* (y) *will pass through* (z).

Demonstration.—If we write the equation of the cubic in the form

$$a_x^3 = 0,$$

then the condition that the polar line of (y) with respect to the polar conic of (x) will pass through (z) will be given by the equation

$$a_x \,.\, a_y \,.\, a_z = 0;$$

and if this relation holds, it is evident that the three points may be interchanged in any manner: hence the proposition is proved.

21. *Definition.*—We shall call a system of points x, y, z, fulfilling the conditions of the last article, a *conjugate system.* It is evident that the system x, y, z is an extension of the binary system which holds for conics, and that itself may be extended; that, for curves of the n^{th} degree there are conjugate systems of n points.

22. If in a conjugate system of three points one of the points, say (z), move on a given line N, we have the following theorem: *if the point* (y) *be the pole of the line* N *with respect to the polar conic of* (x), *then the point* (x) *will be the pole of* N *with respect to the polar conic of* (y).

If we suppose the line N at infinity, we have the following theorem, which is an evident extension of a known theorem in cubics: *if the polar conic of a point* (x) *has its centre at* (y), *then the polar conic of* (y) *has its centre at* (x).

23. If two points of a conjugate system move on the lines λ_x and μ_x, it is not difficult to prove that the locus of the third point is the determinant—

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \lambda_1, \\ u_{21}, & u_{22}, & u_{23}, & \lambda_2, \\ u_{31}, & u_{32}, & u_{33}, & \lambda_3, \\ \mu_1, & \mu_2, & \mu_3, & \end{vmatrix} = 0.$$

Again, if two points of a conjugate system coincide and move on the line λ_x, the locus of the third point will be the curve $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix} = 0.$

24. We shall now solve the question: To find the conditions that two of the points of a conjugate system, say (y) and (z), may be conjugate points with respect to a given conic $(a, b, c, f, g, h)(x_1, x_2, x_3)^2$?

The conditions of the problem evidently give us the following identities—

$$y_1u_{11} + y_2u_{12} + y_3u_{13} = k(ay_1 + hy_2 + gy_3),$$
$$y_1u_{21} + y_2u_{22} + y_3u_{33} = k(hy_1 + hy_2 + fy_3),$$
$$y_1u_{31} + y_2u_{32} + y_3u_{33} = k(gy_1 + fy_2 + cy_3).$$

Hence, eliminating y_1, y_2, y_3, we have the following determinant—

$$\begin{vmatrix} u_{11} - ka, & u_{12} - kh, & u_{13} - kg, \\ u_{21} - kh, & u_{22} - kh, & u_{23} - kf, \\ u_{31} - kg, & u_{32} - kf, & u_{33} - kc, \end{vmatrix} = 0. \qquad (22)$$

This is the Hessian of the curve $U - kS$.

25. If we denote the fundamental cubic by a_y^3, then its polar conic, with respect to the point x, is $a_y^2 \,.\, a_x$, and equation (22) is the condition that $a_y^2 \,.\, a_x - kS$ may represent a pair of lines. We find as follows the locus of the double points of this pair. The equations of the last article may evidently be written

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = kS_1,$$
$$u_{21}x_1 + u_{22}x_2 + u_{23}x_3 = kS_2,$$
$$u_{31}x_1 + u_{32}x_2 + u_{33}x_3 = kS_3,$$

where S_1, S_2, S_3 denote the differentials of S with respect to the variables x_1, x_2, x_3. Hence, if the minors of the Hessian be written A_{11}, A_{12}, &c., we have the values of x_1, x_2, x_3 given by the following equations:—

$$Hx_1 = k(S_1A_{11} + S_2A_{12} + S_3A_{13}),$$
$$Hx_2 = k(S_1A_{21} + S_2A_{22} + S_3A_{23}),$$
$$Hx_3 = k(S_1A_{31} + S_2A_{32} + S_3A_{33}).$$

If these values be substituted in the determinant (22), and if we put the

symbolical expression $S_1A_1 + S_2A_2 + S_3A_3 = S_A$, we get the following determinant—

$$\begin{vmatrix} (S_A)A_1 \;\; - aH, & (S_A) \,.\, mA_3 - hH, & (S_A) \,.\, mA_2 - gH, \\ (S_A)mA_3 - hH, & (S_A) \,.\, A_2 \;\; - bH, & (S_A) \,.\, mA_1 - fH, \\ (S_A)mA_2 - gH, & (S_A)mA_1 \;\; - fH, & (S_A) \,.\, A_3 \;\; - cH, \end{vmatrix} = 0. \quad (23)$$

In the expansion of this determinant we are to substitute for the products $A_1 \,.\, A_2$, $A_1 \,.\, A_3$, &c., which occur, the minors A_{12}, A_{13}, &c., of the Hessian, and we see that the locus of the double points of the line pairs of $\alpha_y{}^2 \,.\, \alpha_x - kS$, is a curve of the ninth degree.

26. If the conic $\alpha_y{}^2 \,.\, \alpha_x$ touches the conic S, it is evident the double point of the line pair of $\alpha_y{}^2 \,.\, \alpha_x - kS$, will be the point of contact, and the determinant (22) will have two equal roots or two equal values for k: hence the discriminant of that determinant, considered as a cubic in k, will be the locus of points whose polar conics touch a given conic.

27. Let the curve U be of the n^{th} degree, and, expanding the determinant (22), let it be

$$H - Ek + \Sigma k^2 - \Delta k^3 = 0. \quad (24)$$

In this equation H is the Hessian of U, E is the equation (4) of Art. 6, and Σ is the locus of points whose polar conics are conjugate to S: the degrees of H, E, Σ are $3(n-2)$, $2(n-2)$, $(n-2)$. Now, the discriminant of the equation (24) is

$$27H^2\Delta^2 + 4H\Sigma^3 + 4\Delta E^3 - E^2\Sigma^2 - 18\Delta HE\Sigma, \quad (25)$$

which represents a curve of the degree $6(n-2)$.

28. If we denote the equation (25) by ∇, and putting $H = 0$, it reduces to

$$E^2(4\Delta E - \Sigma^2)\,; \quad (26)$$

hence ∇ touches the Hessian in all the points where E meets the Hessian. Now, E being of the degree $2(n-2)$ meets the Hessian in $6(n-2)^2$ points, and we shall prove in the next chapter that these correspond to the

$6(n-2)^2$ points in which the conic S meets the Steinerian of U. The remaining points in which ∇ meets the Hessian lie on the curve $4\Delta E - \Sigma^2 = 0$.

29. The curve ∇ may be written in the form

$$(2\Sigma^3 - 9\Delta\Sigma E + 27\Delta^2 H)^2 = 4(\Sigma^2 - 3\Delta E)^3. \tag{27}$$

Hence, ∇ has $6(n-2)^2$ cusps which lie on the curve

$$\Sigma^2 - 3\Delta E. \tag{28}$$

The following are the characteristics of this curve:—

$$\mu = 6(n-2).$$
$$k = 6(n-2)^2.$$
$$\delta = 0.$$
$$i = 60(n-2)^2 - 36(n-2).$$
$$\tau = 27\{3(n-2)^2 - 2(n-2)\}\{2(n-2)^2 - 1\}.$$
$$\nu = 18(n-2)^2 - 6(n-2).$$
$$\text{Deficiency} = 12n^2 - 57n + 67.$$

30. When $n = 3$, the curve ∇ is the 12th class of the 6th degree, and has six cusps which lie on a conic, and the proposition we have solved may be stated as follows: *If two of three conjugate points with respect to a given cubic be conjugate points on a given conic, find the locus of the third point.* We shall, in a future chapter, solve the more important question: If one of three points of a conjugate system moves along a given conic, and the other two coincide, find the equation of the curve they describe.

Again, we shall suppose one of the points to describe a given line, the second a given conic, and we shall find the locus of the third point.

CHAPTER II.

Section I.—Hessian Transformation.

31. If we denote the minors of the determinant which represents the Hessian of a curve of the n^{th} degree by A_{11}, A_{12}, &c., and substitute for the variables x_1, x_2, x_3 of any ternary quantic the symbols A_1, A_2, A_3, which, by convention, are merely umbral (see Salmon's *Algebra*, page 267; also Clebsch's *Theorie der Binaren Algebraischen Formen*), but their squares and products giving the interpretable results A_{11}, A_{12}, &c., that is the minors of the determinant, we shall get a result which I have called the ***Hessian Transformation*** of the given quantic. Thus, if the given quantic be $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^n$, or $\alpha_x{}^n$, its Hessian transformation will be $(\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3)^n$, or $\alpha_A{}^n$. It is evident that in this transformation, if the curve to be transformed be of odd degree, we must square it before making our substitutions. The method will be found to yield important results. In the application of it, the following theorem will be useful.

32. If a be any point on the Hessian of a fundamental curve U of the n^{th} degree, it is required to find the co-ordinates of the point on the Steinerian which corresponds to the point a on the Hessian.

The polar conic of a is

$$(u_{11},\ u_{22},\ u_{33},\ u_{23},\ u_{31},\ u_{12})(x_1,\ x_2,\ x_3)^2,$$

where u_{11}, u_{22}, &c., denote the results of substituting a_1, a_2, a_3, the co-ordinates of a in second differentials; and, since this polar conic denotes two lines intersecting in the point required, we find by an obvious method the co-ordinates to be proportional to the quantities A_{11}, A_{12}, A_{13}, where these are the results of substituting a_1, a_2, a_3 in the minors of the Hessian. Again, since a_1, a_2, a_3 are the co-ordinates of a point on the Hessian, we have

$$A_{11} \cdot A_{22} = A_{12}^2,$$
$$A_{22} \cdot A_{33} = A_{23}^2,$$
$$A_{33} \cdot A_{11} = A_{31}^2.$$

Hence, A_{11}, A_{12}, A_{13} are proportional to the square roots $\sqrt{A_{11}}$, $\sqrt{A_{22}}$, $\sqrt{A_{33}}$, or symbolically to A_1, A_2, A_3; hence we have the following important theorem:—

The Hessian transformation of any point on the Hessian of any curve is the corresponding point on the Steinerian of the same curve; and, conversely, the Hessian transformation of any point on the Steinerian is the corresponding point on the Hessian.

33. From the last Article we have the following theorem:—*The Hessian transformation of the Hessian of any curve of even degree is the Steinerian of the same curve, and the Hessian transformation of any curve of odd degree contains the Steinerian as a factor.*

Thus, if the curve U be of even degree (n), its Hessian, which is of degree $3(n-2)$, will be of even degree, and the substitutions of the minors, which are of degree $2(n-2)$, for the squares and products of the variables, will give a result of degree $3(n-2)^2$, and this is the degree of the Steinerian.

Secondly, if U be of odd degree, since we must square its Hessian before applying our rules, we get a result of the degree $6(n-2)^2$, which we see from the last Article must contain the Steinerian as a factor.

Let U be the quartic

$$ax_1^4 + bx_2^4 + cx_3^4 + 6fx_2^2x_3^2 + 6gx_3^2x_1^2 + 6hx_1^2x_2^2,$$

the Hessian is the determinant

$$\begin{vmatrix} ax_1^2 + hx_2^2 + gx_3^2, & 4hx_1x_2, & 4gx_3x_1 \\ 4hx_1x_2, & hx_1^2 + bx_2^2 + fx_3^2, & 4fx_2x_3 \\ 4gx_3x_1, & 4fx_2x_3, & gx_1^2 + fx_2^2 + cx_3^2 \end{vmatrix}; \quad (29)$$

hence the equation of the Steinerian is

$$\begin{vmatrix} aA_{11} + hA_{22} + gA_{33}, & 4hA_{12}, & 4gA_{31} \\ 4hA_{12}, & hA_{11} + bA_{22} + fA_{33}, & 4fA_{23} \\ 4gA_{31}, & 4fA_{23}, & gA_{11} + fA_{22} + cA_{33} \end{vmatrix}; \quad (30)$$

or expanded,

$$
\begin{aligned}
& agh A_{11}^3 + bhf A_{22}^3 + cfg A_{33}^3 \\
& + (abg + gh^2 + haf) A_{11}^2 . A_{22} + (abf + fh^2 + ghb) A_{11} . A_{22}^2 \\
& + (bch + hf^2 + fgb) A_{22}^2 . A_{33} + (bcg + gf^2 + hfc) A_{22} . A_{33}^2 \\
& + (caf + fg^2 + ghc) A_{33}^2 . A_{11} + (cah + hg^2 + fga) A_{33} . A_{11}^2 \\
& + (abc + 130 fgh + af^2 + bg^2 + ch^2) A_{11} . A_{22} . A_{33} \\
& - 16 f^2 (a A_{11} + h A_{22} + g A_{33}) A_{23}^2 - 16 g^2 (h A_{11} + b A_{22} f A_{33}) A_{31}^2 \\
& - 16 h^2 (g A_{11} + f A_{22} + c A_{33}) A_{12}^2 = 0, \qquad (31)
\end{aligned}
$$

where A_{11}, A_{12}, &c., are the minors of the determinant (29); and, since these are of the 4th degree, this equation will be of the 12th, as it ought.

34. *If any curve of even degree cut the Steinerian, the transformed curve will cut the Hessian in the corresponding points.*

Demonstration.—Let the curve V to be transformed be of the degree $2m$, then this considered as a function of the squares and products of the variables will be of the degree m; and, since the curve U, whose Hessian we use in transformation, is of the n^{th} degree, the minors A_{11}, A_{22}, &c., will be of the degree $2(n-2)$; hence, if W be the transformed of V, the degree of W will be $2m(n-2)$; therefore, the number of points of intersection of W and the Hessian will be $6m(n-2)^2$. Again, the Steinerian of U is of the degree $3(n-2)^2$; hence the number of points of intersection of V with the Steinerian is equal to the number of points of intersection of W with the Hessian.

35. If V be of odd degree, since we must square it before applying the transformation, we have the following theorem:—

If any curve V of odd degree cuts the Steinerian of a fundamental curve U in any number of points, the transformed of V will touch the Hessian of U in the corresponding points.

36. Since the Steinerian of a curve of the third order is also its Hessian, we have, from the theorems of the last two articles, the following theorems:—

1°. *If any curve of even degree cuts the Hessian of a cubic in any number of points, the transformed of the curve will cut the Hessian in the corresponding points.*

2°. *If any curve of odd degree cut the Hessian in any number of points, the transformed of that curve will touch the Hessian in the corresponding points.*

37. We shall now give some applications of Hessian transformations, and others will occur in the sequel:—

Find the Hessian transformation of λ_x.

Squaring, and substituting as in recent Articles, we get the result already given in equation (6), Art. 6; hence, *the Hessian transformation of λ_x is the "Zwischenform" E*; and we see from Art. 35 that E touches the Hessian of the fundamental curve U in the points corresponding to the points of intersection of λ_x with the Steinerian of U; hence we have an independent proof of the theorem of Art. 11, namely, *that the curve there represented by L touches the Hessian everywhere it meets it.*

38. The transformation of the product of the lines λx, μx is the determinant (5), Art. 6. This curve may be written in the form

$$\mu_1 \frac{dE}{d\lambda_1} + \mu_2 \frac{dE}{d\lambda_2} + \mu_3 \frac{dE}{d\lambda_3} = 0; \tag{32}$$

and it intersects the Hessian in all the points which correspond to the points of intersection of λ_x and μ_x with the Steinerian.

Cor. 1.—*The differentials* $\frac{dE}{d\lambda_1}$, $\frac{dE}{d\lambda_2}$, $\frac{dE}{d\lambda_3}$, *are the transformations of the products of the line λ_x, and the three sides, x_1, x_2, x_3, respectively, of the triangle of reference.*

Cor. 2.—*The curves* $\frac{dE}{d\lambda_1}$, $\frac{dE}{d\lambda_2}$, $\frac{dE}{d\lambda_3}$, *have common all the points of the Hessian which correspond to the points of intersection of λ_x with the Steinerian, that is, $3(n-2)^2$ points. When U is a cubic, these three differentials will be conics, and they will have three common points forming a triangle.*

We shall, in a future Article, find the equation of the sides of this triangle. It will be a "Zwischenform."

39. The Hessian transformation of a line which passes through a given point may be found thus: Let λ_x and μ_x be two lines through the given point, then any variable line through the given point will be represented by $\lambda_x + k\mu_x$; then squaring and substituting, we get the required transformation, $L + 2kR + k^2M = 0$, where L and M are the transformations of λ_x, and μ_x and R is the transformation of their product.

40. The Hessian transformation of the conic a_x^2 is the curve a_A^2 of degree $2(n-2) = 0$. This is the curve E represented by equation (4), Art. 6; hence we have the following theorem: *The Hessian transformation of any conic is the locus of points to whose polar conic with respect to any curve U of degree n the given conic is conjugate, and we have a proof of the theorem stated in Art.* 28, *that E meets the Hessian in the points which correspond to the points of intersection of the conic a_x^2 with the Steinerian.*

Cor.—When the fundamental curve is a cubic, the Hessian transformation a_A^2 is a conic; hence we have Cremona's theorem: "*If a conic intersect a cubic in six points, the six corresponding points will lie on another conic.*"—Cremona's *Plane Curves*, Art. 45; Durege, *Curven Dritter Ordnung*, Art. 234.

41. Since from any point six tangents can be drawn to touch a given cubic, and since the six points of contact lie on a given conic, and the other six points in which they meet the cubic lie on another conic; from the theorems of recent articles we infer the following theorem: *Six conics can be described to touch a given line, and having double contact with a given cubic, the contacts being respectively two-pointic and four-pointic, and the six four-pointic and six two-pointic contacts lie respectively on two conics.*

42. *If the fundamental curve U be a cubic, the Hessian transformation of the polar conic of any point with respect to the cubic is the polar conic of the same point with respect to the Hessian.*

Demonstration.—Let $U = x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3$; hence, omitting the factor 3,

$$\frac{dU}{dx_1} = x_1^2 + 2mx_2x_3;$$

hence, the transformed of $\frac{dU}{dx_1}$ is

$$A_{11} + 2mA_{23};$$

and restoring the values of A_{11}, A_{23}, we find the transformed of $\frac{dU}{dx_1}$ is $\frac{dH}{dx_1}$; and similarly for the other differentials; hence the proposition is proved.

Cor.—The Hessian transformation of the polar conic of any point with respect to the Hessian of a cubic is the polar conic of the same point with respect to the cubic.

43. If the fundamental curve U be a cubic, we have $3U = x_1U_1 + x_2U_2 + x_3U_3$, where U_1, U_2, U_3 are differential coefficients; hence, the Hessian transformation of U^2 is

$$(A_1H_1 + A_2H_2 + A_3H_3)^2;$$

or the determinant,

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & H_1, \\ u_{21}, & u_{22}, & u_{23}, & H_2, \\ u_{31}, & u_{32}, & u_{33}, & H_3, \\ H_1, & H_2, & H_3, & \end{vmatrix} = 0. \qquad (33)$$

This covariant is given in Salmon's *Higher Curves*, Art. 231, page 195; and we infer from the property of the Hessian transformation that it touches the Hessian in the nine points which correspond to the nine points of inflection.

Cor.—If we use Dr. Salmon's notation for the covariants, the covariant (33)

$$\equiv -4(\Theta + 3SUH); \qquad (34)$$

hence we have the following theorem: *Dr. Salmon's Θ touches the Hessian in the nine points which correspond to the nine points of inflection.*

44. In the same manner the transformation of H^2 is

$$(A_1U_1 + A_2U_2 + A_3U_3)^2;$$

or, omitting numerical factors, the Hessian transformation of H^2 is

$$(HU):$$

(see Salmon's *Modern Algebra*, page 17).

45. Let D be a point on the Hessian, C its corresponding point; then, since the polar conic of D with respect to the fundamental cubic consists of two lines intersecting in C, let these lines be λ_x and μ_x; then the Hessian transformation of the product of these lines is the conic $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$; and this, by Art. 42, is the polar conic of D with respect to the Hessian; and it, by Art. 11, passes through the points of contact of H and the Hessian transformation of λ_x and μ_x; hence we have the following theorem: *If from any point D on the Hessian tangents be drawn, the polar conic of D, with respect to the cubic, consists of one of the three pairs of lines, namely, λ_x, μ_x, joining the points of contact.*

Cor.—From this proposition we have an independent proof that the three pairs of lines which join the points of contact of four tangents from any point on a cubic intersect on the cubic, the points of intersection being the three points which correspond, with respect to the three cubics which have the given curve as a Hessian, to the point whence the tangents are drawn.

46. If the lines λ_x, μ_x, be tangents to a cubic at corresponding points, then the Hessian transformation $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ of their product will be a perfect square, namely, the square of the line joining the corresponding points.

47. *The Hessian transformation $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$ of the line λ_x meets λ_x in two points which represent the Hessian of the cubic whose roots are represented by the*

distances from any arbitrary point on that line to the three points in which it meets the fundamental cubic.

Demonstration.—Let the three points in which λ_x meets the cubic be A, B, C, and the two points in which it intersects $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$ be D and E; then, since the conic $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$ is the locus of points whose polar conics with respect to the fundamental cubic touch λ_x, the polars of the points D and E with respect to the system of three points, A, B, C, will have double points; and consequently these double points will, by the theory of binary cubics, be the points D and E: hence the proposition is proved. (See Clebsch's *Theorie der Algebraischen Binaren Formen.*)

48. We have seen, Art. 5, equation (3), that the Hessian of $U + kL^3 = 0$, where L is the line λ_x, is

$$H + kLE,$$

where E is the Hessian transformation of λ_x with respect to the fundamental cubic; hence, the Hessian of $U + kL^3$ touches H in the three points where E touches H. Again, the transformation of λ_x with respect to the Hessian of $U + kL^3$ must touch the latter in the same three points; hence we have the following theorem: *The transformations of a line λ_x, with respect to the Hessians of all cubics, having the same line for a chord of osculation, are identical.*

This theorem may be proved directly as follows: The Hessian transformation of λ_x, if the fundamental cubic be $U + kL^3$, is the determinant

$$\begin{vmatrix} u_{11} + k\lambda_1^2 L, & u_{12} + k\lambda_1\lambda_2 L, & u_{13} + k\lambda_1\lambda_3 L, & \lambda_1 \\ u_{21} + k\lambda_2\lambda_1 L, & u_{22} + k\lambda_2^2 L, & u_{23} + k\lambda_2\lambda_3 L, & \lambda_2 \\ u_{31} + k\lambda_3\lambda_1 L, & u_{32} + k\lambda_3\lambda_2 L, & u_{33} + k\lambda_3\lambda_1 L, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3 & \end{vmatrix};$$

and, expanding, it will be found that the terms involving k and k^2 vanish identically.

49. If U and $U + M^3$ be two cubics ($M \equiv \mu_x$), then the transformation of λ_x with respect to the Hessians of these cubics,

$$E \text{ and } E + MN,$$

where $E = \binom{\lambda}{\lambda}$, and N is the polar with respect to U of the point of intersection of the lines λ_x and μ_x; hence we have the following theorem: *If a system of cubics have a common chord of osculation, the transformations of a given line with respect to the Hessians of this system pass through four common points.*

50. Since two conics of the system

$$E + kMN$$

can be described to touch a given line, it follows, from Art. 47, that two cubics of the system $U + kM^3$ can be described to touch a given line; or this may be proved directly as follows: The given line intersects U in three points, which will be determined by a cubic equation, and it will intersect $U + kM^3$ in three other points, which will be determined by a cubic equation differing from the latter only in the absolute term, and this absolute term is of the second degree in the discriminant of the cubic: hence the proposition is proved.

Section II.—Cayleyan Transformation.

51. The Hessian transformation of the conic $(a, b, c, f, g, h)(x_1, x_2, x_3)^2$, is

$$\begin{aligned}&(m^2a + 2mf)x_1^2 + (m^2b + 2mg)x_2^2 + (m^2c + 2mh)x_3^2 \\ &- (a + 2m^2f)x_2x_3 - (b + 2m^2g)x_3x_1 - (c + 2m^2h)x_1x_2 = 0.\end{aligned}$$

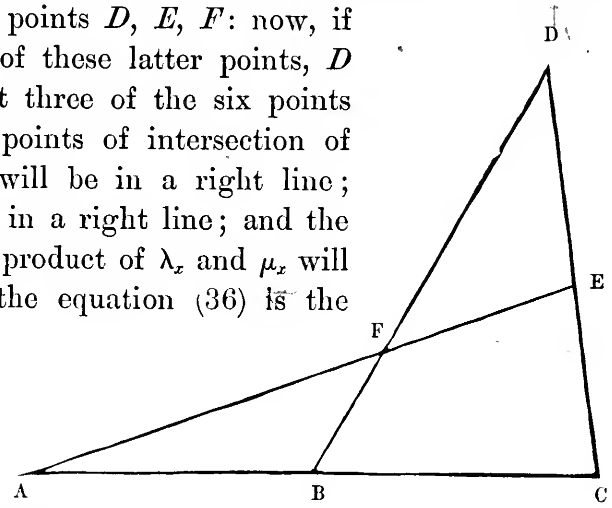
The discriminant of this conic is

$$\begin{aligned}&m^2(a^3 + b^3 + c^3) + (1 - 4m^6)abc + 4m^3(f^3 + g^3 + h^3) + 8m^6fgh \\ &+ 2m(1 + 2m^3)(a^2f + b^2g + c^2h) + 4m^3(2 + m^3)(af^2 + bg^2 + ch^2) \\ &+ 2m^2(1 - 4m^3)abc\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right) - 12m^4fgh\left(\frac{a}{f} + \frac{b}{g} + \frac{c}{h}\right). \qquad (35)\end{aligned}$$

52. If the conic of the last article be the product of two lines λ_x, μ_x, we get the discriminant of the Hessian transformation of this product by substituting in equation (35) for the coefficients a, b, c, &c., of the conic, the products $\lambda_1\mu_1$, $\lambda_2\mu_2$, $\lambda_3\mu_3$, &c.: hence the discriminant of the product of λ_x and μ_x is

$$
\begin{aligned}
& m^2\{\lambda_1^3\mu_1^3 + \lambda_2^3\mu_2^3 + \lambda_3^3\mu_3^3\} + (1 - 4m^6)\lambda_1\lambda_2\lambda_3\mu_1\mu_2\mu_3 \\
& + \frac{m^5}{2}\left\{(\lambda_1\mu_2 + \lambda_2\mu_1)^3 + (\lambda_2\mu_3 + \lambda_3\mu_2)^3 + (\lambda_3\mu_1 + \lambda_1\mu_3)^6\right\} \\
& + m^6\{(\lambda_1\mu_2 + \lambda_2\mu_1)(\lambda_2\mu_3 + \lambda_3\mu_2)(\lambda_3\mu_1 + \lambda_1\mu_3) \\
& + m(1 + 2m^3)\{\lambda_1^2\mu_1^2(\lambda_2\mu_3\lambda_3\mu_2) + \lambda_2^2\mu_2^2(\lambda_3\mu_1 + \lambda_1\mu_3) + \lambda_3^2\mu_3^2(\lambda_1\mu_2 + \lambda_2\mu_1)\} \\
& + m^3(2 + m^3)\{\lambda_1\mu_1(\lambda_2\mu_3 + \lambda_3\mu_2)^2 + \lambda_2\mu_2(\lambda_3\mu_1 + \lambda_1\mu_3)^2 + \lambda_3\mu_3(\lambda_1\mu_2 + \lambda_2\mu_1)^2\} \\
& + m^2(1 - 4m^3)(\lambda_1\lambda_2\lambda_3\mu_1\mu_2\mu_3)\left\{\frac{\lambda_1\mu_2 + \lambda_2\mu_1}{\lambda_3\mu_3} + \frac{\lambda_2\mu_3 + \lambda_3\mu_1}{\lambda_1\mu_1} + \frac{\lambda_3\mu_1 + \lambda_1\mu_3}{\lambda_2\mu_2}\right\} \\
& - \frac{3m^4}{2}\left(\lambda_1\mu_2 + \lambda_2\mu_1)(\lambda_2\mu_3 + \lambda_3\mu_1)(\lambda_3\mu_1 + \lambda_1\mu_3\right) \\
& \times\left\{\frac{\lambda_1\mu_1}{\lambda_2\mu_3 + \lambda_3\mu_2} + \frac{\lambda_2\mu_2}{\lambda_3\mu_1 + \lambda_1\mu_3} + \frac{\lambda_3\mu_3}{\lambda_1\mu_2 + \lambda_2\mu_1}\right\}.
\end{aligned} \tag{36}
$$

53. Let the points in which the line λ_x meets the Hessian be A, B, C, and their three corresponding points D, E, F: now, if the line μ_x pass through one of these latter points, D for example, it is evident that three of the six points which correspond to the six points of intersection of λ_x and μ_x with the Hessian, will be in a right line; hence the other three will be in a right line; and the Hessian transformation of the product of λ_x and μ_x will consist of two lines: hence, the equation (36) is the tangential equation in μ variables of the three points which correspond to the points of intersection of the line λ_x with the Hessian.



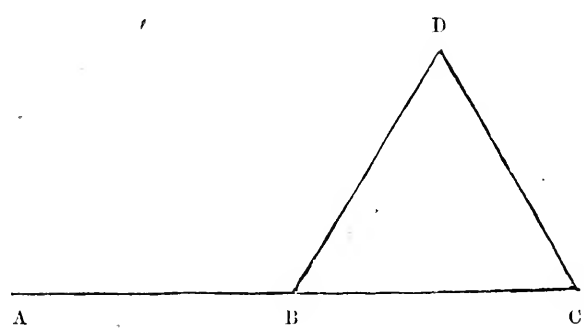
54. If in equation (36) we replace μ_1, μ_2, μ_3 by λ_1, λ_2, λ_3, we get, after an easy reduction,

$$- 2 \{m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3\}^2; \qquad (37)$$

and this is the condition that the line λ_x shall pass through two corresponding points on the Hessian; in other words, it is the envelope of lines which join two corresponding points.

The contravariant $m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3$, to which we have been just led, is called the Cayleyan of the cubic, after Professor Cayley, who first directed the attention of mathematicians to its great importance, and, from our investigations, we have the following theorem: *If a line be a tangent to the Cayleyan, its Hessian transformation consists of a pair of lines intersecting on and touching the Hessian* (see Art. 35).

55. If B and C be corresponding points on the Hessian, A the third point in which the line joining the points B, C, meets the Hessian, BD, CD tangents at B and D, then D is the point which corresponds to A, and the Hessian transformation of the equation of BD will be a conic touching the Hessian in the points A and C, and having two pointic contact at A and four pointic contact at C. In like manner, the equation of CD gives, by Hessian transformation, a conic having two pointic contact at A, and four pointic contact at B: hence we have the following remarkable theorem: *If a variable conic has double contact with the Hessian, the contacts being two pointic and four pointic respectively, the envelope of the chord of contact is the Cayleyan.*



Cor.—Since every cubic is the Hessian of some other cubic, we have the following theorem: *If a variable conic has double contact with a cubic, the contacts being respectively two pointic and four pointic, the envelope of the chord of contact is a curve of the third class.*

56. Since from any point on a cubic four tangents can be drawn which will touch the cubic elsewhere, we have the following theorem: *Four conics can be described having two pointic contact with a cubic at any point and four pointic contact at other points.*

57. Since any point on a cubic being given, three other points can be found, each of which may be considered as the corresponding point, as we regard it as the Hessian of any of three different cubics: *hence, through any point A on a cubic can be described three conics touching the cubic elsewhere, and having four pointic contact at A.*

58. The equation of the Cayleyan,

$$m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3 = 0,$$

may be written in the following form as a determinant:

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2 \\ \lambda_3, & -2m\lambda^2, & \lambda_1 \\ \lambda_2, & \lambda_1, & -2m\lambda_3 \end{vmatrix} = 0. \tag{38}$$

We shall denote this contravariant by K, and its minors by the symbols κ_{11}, κ_{12}, κ_{13}, &c., thus:—

$$\begin{aligned} \kappa_{11} &= 4m^2\lambda_2\lambda_3 - \lambda_1^2, \\ \kappa_{22} &= 4m^2\lambda_3\lambda_1 - \lambda_2^2, \\ \kappa_{33} &= 4m^2\lambda_1\lambda_2 - \lambda_3^2, \\ \kappa_{23} &= \lambda_2\lambda_3 + 2m\lambda_1^2, \\ \kappa_{31} &= \lambda_3\lambda_1 + 2m\lambda_2^2, \\ \kappa_{12} &= \lambda_1\lambda_2 + 2m\lambda_3^2. \end{aligned} \tag{39}$$

59. We infer, from Art. 54, that if the line λ_x be a tangent to the

Cayleyan, that its Hessian transformation vanishes. The discriminant of this transformation is the determinant

$$\left|\begin{array}{lll} 2(m^2\lambda_1^2 + 2m\lambda_2\lambda_3), & -(\lambda_3^2 + 2m^2\lambda_1\lambda_2), & -(\lambda_2^2 + 2m^2\lambda_3\lambda_1) \\ -(\lambda_3^2 + 2m^2\lambda_1\lambda_2), & 2(m^2\lambda_2^2 + 2m\lambda_3\lambda_1), & -(\lambda_1^2 + 2m^2\lambda_2\lambda_3) \\ -(\lambda_2^2 + 2m^2\lambda_3\lambda_1), & -(\lambda_1^2 + 2m^2\lambda_2\lambda_3), & 2(m^2\lambda_3^2 + 2m\lambda_1\lambda_2) \end{array}\right| . \qquad (40)$$

Denoting the minors of this determinant by ϵ_{11}, ϵ_{22}, ϵ_{33}, ϵ_{23}, ϵ_{31}, ϵ_{12} respectively, we get, after an easy reduction, the following system of values:—

$$\begin{aligned} \epsilon_{11} &= 8m^2\lambda_1 K + (1 + 8m^3)\lambda_1^2\kappa_{11}, \\ \epsilon_{22} &= 8m^2\lambda_2 K + (1 + 8m^3)\lambda_2^2\kappa_{22}, \\ \epsilon_{33} &= 8m^2\lambda_3 K + (1 + 8m^3)\lambda_3^2\kappa_{33}, \\ \epsilon_{23} &= 2m\lambda_1 K + (1 + 8m^3)\lambda_2\lambda_3\kappa_{23}, \\ \epsilon_{31} &= 2m\lambda_2 K + (1 + 8m^3)\lambda_3\lambda_1\kappa_{31}, \\ \epsilon_{12} &= 2m\lambda_3 K + (1 + 8m^3)\lambda_1\lambda_2\kappa_{12}. \end{aligned} \qquad (41)$$

60. The equation $KU - S(\mu_1x_1 + \mu_2x_2 + \mu_3x_3)^3$, where K denotes the result of substituting the line co-ordinates μ_1, μ_2 μ_3 in the equation of the Cayleyan, and S the quartic invariant $m - m^4$ of U (see *Salmon's Higher Curves*, page 190, Art. 219) will occupy a prominent place in this Memoir. We shall find its Cayleyan here:—

For the equation of the cubic

$$\begin{aligned} ax_1^3 + bx_2^3 + cx_3^3 + 3a_2x_1^2x_2 + 3a_3x_1^2x_3 + 3_1x_1x_2^2 + 3b_3x_2^2x_3 \\ + 3c_1x_1x_3^2 + 3c_2x_2x_3^2 + 6mx_1x_2x_3 = 0, \end{aligned}$$

Dr. Salmon gives (*Higher Curves*, page 183),

$$(A,\ B,\ C,\ A_2,\ A_3,\ B_1,\ B_3,\ C_1,\ C_2\ M)(\lambda_1\ \lambda_2\ \lambda_3) = 0\ ;$$

where

$$A = bcm - bc_1c_2 - mb_3c_2 + b_1c_2^2 + c_1b_3^2 - cb_1b_3,$$

$$B = cam - ca_2a_3 - ma_3c_1 + a_2c_1^2 + c_2a_3^2 - ac_1c_2,$$

&c., &c., as the equation of the Cayleyan.

Hence, comparing our equation with Dr. Salmon's, we get the following values :—

$$
\begin{aligned}
A &= K^2\,\{m^2\mu_1^3 + m^5(\mu_2^3 + \mu_3^3) - 3m^4\mu_1\mu_2\mu_3\},\\
B &= K^2\,\{m^2\mu_2^3 + m^5(\mu_3^3 + \mu_1^3) - 3m^4\mu_1\mu_2\mu_3\},\\
C &= K^2\,\{m^2\mu_3^3 + m^5(\mu_1^3 + \mu_2^3) - 3m^4\mu_1\mu_2\mu_3\},\\
3A_2 &= K^2S\,\{\mu_1^2\mu_3 + m\mu_1\mu_2^2 - 2m^2\mu_2\mu_3^2\},\\
3A_3 &= K^2S\,\{\mu_1^2\mu_2 + m\mu_1\mu_3^2 - 2m^2\mu_2^2\mu_3\},\\
3B_1 &= K^2S\,\{\mu_2^2\mu_3 + m\mu_1^2\mu_2 - 2m^2\mu_1\mu_3^2\},\\
3B_3 &= K^2S\,\{\mu_1\mu_2^2 + m\mu_2\mu_3^2 - 2m^2\mu_1^2\mu_3\},\\
3C_1 &= K^2S\,\{\mu_2\mu_3^2 + m\mu_1^2\mu_3 - 2m^2\mu_1\mu_2^2\},\\
3C_2 &= K^2S\,\{\mu_1\mu_3^2 + m\mu_2^2\mu_3 - 2m^2\mu_1^2\mu_2\},\\
6M &= K\,\{1 + 2m^3)^2\,\mu_1\mu_2\mu_3 - 3m^4(\mu_1^3 + \mu_2^3 + \mu_3^3)\}.
\end{aligned}
$$

Hence, substituting these values and comparing with the result of Art. 52, we see that the latter is equal to the former multipled by

$$K^2 \text{ or by } \{m(\mu_1^3 + \mu_2^3 + \mu_3^3) + (1 - 4m^3)\,\mu_1\mu_2\mu_3\}^2.$$

Hence we have the following theorem :—

Considering the "Zwischenform"

$$KU - S(\mu_1x_1 + \mu_2x_2 + \mu_3x_3)^3 = 0$$

as the equation of a curve in point co-ordinates, the equation of its Cayleyan is equal to K^2 *multiplied by the discriminant of the Hessian transformation of the product of the lines* λ_x *and* μ_x, *that is, equal to* K^2 *multiplied by the discriminant of the conic* $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$.

61. In the same manner as we showed in Art. 31 how to infer from the equation of any curve given in point co-ordinates the equation of

another curve called its Hessian transformation, so we can, by means of the equations (39) of Art. 58, infer from the equation of any curve in line co-ordinates the equation of another curve also in line co-ordinates, and which we shall call the Cayleyan transformation of the former curve. Thus, if λ_1, λ_2, λ_3 be line co-ordinates, for these we are to substitute κ_1, κ_2, κ_3 where these symbols are merely umbral, but for their squares and products we are to substitute the minors of the Cayleyan κ_{11}, κ_{12}, &c.: see equation (39). In this, as in the former transformation, it is evident that if the quantic to be transformed be of odd degree, we must square it before applying our rules, or we may first substitute and then square. All this will have been sufficiently evident from the last section.

62. If x_1, x_2, x_3 be the co-ordinates of any point, its equation is $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, and the Cayleyan transformation of this equation is $(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)^2$; or expanding and substituting for κ_{11}, κ_{12}, &c., their values, we get the determinant

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2, & x_1, \\ \lambda_3, & -2m\lambda_2, & \lambda_1, & x_2, \\ \lambda_2, & \lambda_1, & -2m\lambda_3, & x_3, \\ x_1, & x_2, & x_3 & \end{vmatrix}. \qquad (42)$$

This conic has triple contact with the Cayleyan, the tangents at the points of contact being those which correspond to the three tangents from the point $(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)$ to the Cayleyan. We shall find this "Zwischenform" to be an important one in the theory of cubics, its relation to the Cayleyan being the reciprocal of the relation of the Hessian transformation of a line to the Hessian.

63. We have seen that the Hessian transformations of the differentials of the Hessian give the differentials of the fundamental cubic. We shall now find a curve of third class, which shall have a corresponding relation to the Cayleyan.

We have

$$K \equiv m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3;$$

$$\therefore \quad \frac{dK}{d\lambda_1} \equiv 3m\lambda_1^2 + (1 - 4m^3)\lambda_2\lambda_3.$$

Hence the Cayleyan transformation of $\frac{dK}{d\lambda_1}$ is

$$3mk_{11} + (1 - 4m^3)k_{23}$$

$$= 3m(4m^2\lambda_2\lambda_3 - \lambda_1^2) + (1 - 4m^3)(\lambda_2\lambda_3 + 2m\lambda_1^2$$

$$= (1 + 8m^3)(\lambda_2\lambda_3 - m\lambda_1^2).$$

Again, if we differentiate the curve

$$\Psi \equiv 3\lambda_1\lambda_2\lambda_3 - m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3), \qquad (43)$$

we have

$$\frac{d\Psi}{d\lambda_1} = 3(\lambda_2\lambda_3 - m\lambda_1^2).$$

Hence, omitting numerical factors, we have the following theorem :—

The Cayleyan transformation of the differentials of the Cayleyan give the differentials of the curve Ψ; and conversely, the Cayleyan transformations of the differentials of Ψ give the corresponding differentials of the Cayleyan; and we see that Ψ bears the same relation to the Cayleyan that the fundamental cubic has to the Hessian.

Cor. 1.—*The Cayleyan transformation of a conic S, whose equation is given in line co-ordinates, is the envelope of lines whose polar conics, with respect to the curve Ψ, are conjugate to the conic S.*

Cor. 2.—*The Cayleyan transformation of the product of the equations of two points, or say a point pair, is the envelope of lines to whose polar conics, with respect to Ψ, the points are conjugate.*

Cor. 3.—*The Cayleyan transformation of the equation of a point is the envelope of lines whose polar conics, with respect to Ψ, pass through the point.*

64. If we form the discriminant of the equation (42), with respect to the line variables, we get the equation of the Hessian. Hence we have the following theorem :

The Cayleyan transformation of any point on the Hessian, of the fundamental cubic, consists of two points. The locus of these points is the Cayleyan, and so also is the envelope of their line of connexion.

65. If a variable point move along the line, joining two fixed points x, y, the Cayleyan transformation of the equation of this point will be of the form

$$\Lambda + 2kP + k^2M, \tag{44}$$

where Λ will be the determinant (42), M the same determinant with the co-ordinates of the point (y), substituted for those of x, and P the same determinant bordered vertically with x_s, and horizontally with y_s.

The envelope of the conic (44) will be the product of the Cayleyan and the pole, with respect to the curve Ψ (see Art. 63), of the line joining the points x and y.

66. *The Cayleyan transformation of the equation of a conic Σ is another conic inscribed in the hexagon formed by the six tangents to the Cayleyan, which correspond to the six common tangents of Σ and the Cayleyan.*

This theorem is the analogue of Cremona's theorem: see Art. 40.

Section III.

67. This section will consist of further applications of the Hessian transformation, viz., we shall apply them to certain lines connected with the fundamental cubic, such as the inflectional tangents and lines through the nine points of inflection. We require, for the purposes of reference, the co-ordinates of the nine points of inflection; we accordingly give them

here, and in a future Chapter we shall give the co-ordinates of their corresponding points on the Hessian.

Since the equation, $x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0$, of the fundamental cubic may be written in the form

$$(x_1 + x_2 - 2mx_3)(\omega x_1 + \omega^2 x_2 - 2mx_3)(\omega^2 x_1 + \omega x_2 - 2mx_3) = (1 + 8m^3)x_3^3 \qquad (45)$$

(where ω denotes one of the imaginary cube roots of unity), the points of inflection on x_3 will be its points of intersection with the three lines

$$x_1 + x_2 = 0, \qquad x_1 + \omega x_2 = 0, \qquad \omega x_1 + x_2 = 0,$$

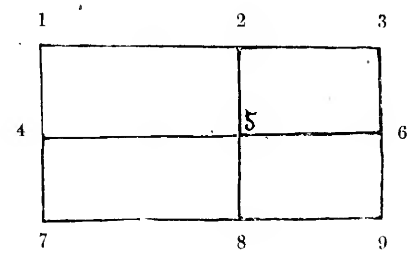
and similarly for the points of inflection on x_1, x_2. In this way we get the co-ordinates in the following Table. Those marked 1), 2), 3), are the points on x_1; 4), 5), 6) are those on x_2, and 7), 8), 9) on x_3. They are marked thus in accordance with the usual notation:—

$$\left.\begin{array}{llll}
1). \quad 0,\ +1,\ -1, & 2). \quad 0,\ \omega,\ -1, & 3). \quad 0,\ 1,\ -\omega \\
4). \quad -1,\ 0,\ +1, & 5). \quad -1,\ 0,\ \omega, & 6). \quad -\omega,\ 0,\ +1 \\
7). \quad +1,\ -1,\ 0, & 8). \quad \omega,\ -1,\ 0, & 9). \quad 1,\ -\omega,\ 0
\end{array}\right\} . \qquad \text{I.}$$

68. The following Table gives the equation of the tangents to the Hessian at the nine points of inflection:—

$$\left.\begin{array}{l}
1). \ (1 + 2m^3)x_1 + 3m^2x_2 + 3m^2x_3 \\
2). \ (1 + 2m^3)x_1 + 3m^2\omega x_2 + 3m^2\omega^2 x_3 \\
3). \ (1 + 2m^3)x_1 + 3m^2\omega^2 x_2 + 3m^2\omega x_3 \\
4). \ 3m^2x_1 + (1 + 2m^3)x_2 + 3m^2x_3 \\
5). \ 3m^2\omega x_1 + (1 + 2m^3)x_2 + 3m^2\omega^2 x_3 \\
6). \ 3m^2\omega^2 x_1 + (1 + 2m^3)x_2 + 3m^2\omega x_3 \\
7). \ 3m^2x_1 + 3m^2x_2 + (1 + 2m^3)x_3 \\
8). \ 3m^2\omega x_1 + 3m^2\omega^2 x_2 + (1 + 2m^3)x_3 \\
9). \ 3m^2\omega^2 x_1 + 3m^2\omega x_2 + (1 + 2m^3)x_3
\end{array}\right\} . \qquad \text{II.}$$

69. The nine points of inflection lie on twelve lines, four of which are real and eight imaginary. The annexed diagram will enable us to form some conception of these lines. If we suppose the diagram drawn on a large sphere, then consider, for instance, the points 8 and 6 joined, and the line of connexion continued round the sphere until it comes to 1: similarly for the others. The following Table contains the twelve lines and their equations:—



$$\left.\begin{array}{ll} (123) = x_1, & (258) = x_1 + \omega x_2 + \omega^2 x_3 \\ (456) = x_2, & (348) = x_1 + \omega x_2 + x_3 \\ (789) = x_3, & (249) = x_1 + \omega^2 x_2 + x_3 \\ (147) = x_1 + x_2 + x_3, & (369) = x_1 + \omega^2 x_2 + \omega x_3 \\ (159) = \omega x_1 + x_2 + x_3, & (267) = x_1 + x_2 + \omega x_3 \\ (168) = \omega^2 x + x_2 + x_3, & (357) = x_1 + x_2 + \omega^2 x_3 \end{array}\right\}. \qquad \text{III.}$$

We can verify that these lines pass through the points stated by means of the system of co-ordinates given in the Table I., Art. 67.

70. The polar conics of the points 1), 2), 3) are $u_2 - u_3$, $\omega^2 u_2 - \omega u_3$, $\omega u_2 - \omega^2 u_3$; and similar forms for the triads 4), 5), 6): 7), 8), 9).

Hence the product of the polar conics of the nine points of inflection is

$$(u_1^3 - u_2^3)(u_2^3 - u_3^3)(u_3^3 - u_1^3) = 0. \tag{46}$$

71. The equations of the nine harmonic polars are

$$\left.\begin{array}{lll} 1).\quad x_2 - x_3, & 2).\quad \omega x_2 - \omega^2 x_3, & 3).\quad \omega^2 x_2 - \omega x_3 \\ 4).\quad x_3 - x_1, & 5).\quad \omega x_3 - \omega^2 x_1, & 6).\quad \omega^2 x_3 - \omega x_1 \\ 7).\quad x_1 - x_2, & 8).\quad \omega x_1 - \omega^2 x_2, & 9).\quad \omega^2 x_1 - \omega x_2 \end{array}\right\}. \qquad \text{IV.}$$

Hence it follows that the three systems (147), (258), (369) are, respectively, concurrent.

Again, the three pencils are equianharmonic which are composed of the lines

$$\begin{array}{lll} x_1 & \text{and} & 7),\ 8),\ 9), \\ x_2 & ,, & 1),\ 2),\ 3), \\ x_3 & ,, & 4),\ 5),\ 6), \end{array}$$

the anharmonic ratio of each pencil being equal to an imaginary cube root of unity (*see* Cremona's *Plane Curves*).

Cor.—The product of the nine harmonic polars

$$= (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3). \tag{47}$$

72. We now return to the Hessian transformation. Since a tangent to the Hessian at a point of inflection meets it in three consecutive points, the Hessian transformation of this tangent will be a conic touching the Hessian in six consecutive points; in other words, the Hessian transformation of an inflectional tangent to the Hessian will be a conic having six pointic contact with it at the point corresponding to the point of inflection; and, since the Hessian being given, there are three cubics of which it is the Hessian, there are in all twenty-seven conics, each touching in six consecutive points.

The Hessian transformation of the first line marked 1), Table II., Article 68, is

$$\begin{vmatrix} x_1, & mx_3, & mx_2, & (1+2m^3) \\ mx_3, & x_2, & mx_1, & 3m^2 \\ mx_2, & mx_1, & x_3, & 3m^2 \\ 1+2m^3, & 3m^2, & 3m^2, & \end{vmatrix} = 0; \tag{48}$$

or expanded,

$$(m^2 + 22m^5 + 4m^8)x_1^2 + 3m^3(1+5m^3)(x_2^2 + x_3^2)$$
$$- (1 + 4m^3 + 22m^6)x_2x_3 - 3m^4(5 + 4m^3)(x_3x_1 + x_1x_2) = 0. \tag{49}$$

73. If we denote the coefficients in the last equation, for shortness, by A, B, C, D respectively, the equations of the Hessian transformation of the system of equations given in Table II., Art. 68, are contained in the following Table: each conic in it has six pointic contact with the Hessian :—

$$\left.\begin{array}{ll}
1). & Ax_1^2 + B(x_2^2 + x_3^2) - Cx_2x_3 - Dx_1(x_2 + x_3) \\
2). & Ax_1^2 + B(\omega^2x_2^2 + \omega x_3^2) - Cx_2x_3 - Dx_1(\omega x_2 + \omega^2 x_2) \\
3). & Ax_1^2 + B(\omega x_2^2 + \omega^2 x_3^2) - Cx_2x_3 - Dx_1(\omega^2 x_2 + \omega x_3) \\
4). & Ax_2^2 + B(x_3^2 + x_1^2) - Cx_3x_1 - Dx_2(x_3 + x_1) \\
5). & Ax_2^2 + B(\omega^2x_3^2 + \omega x_1^2) - Cx_3x_1 - Dx_2(\omega^2 x_3 + \omega x_1) \\
6). & Ax_2^2 + B(\omega x_3^2 + \omega^2 x_1^2) - Cx_3x_1 - Dx_2(\omega x_3 + \omega^2 x_1) \\
7). & Ax_3^2 + B(x_1^2 + x_2^2) - Cx_1x_2 - Dx_3(x_1 + x_2) \\
8). & Ax_3^2 + B(\omega^2x_1^2 + \omega x_2^2) - C(x_1x_2) - Dx_3(\omega^2 x_2 + \omega x_2) \\
9). & Ax_3^2 + B(\omega x_1^2 + \omega^2 x_2^2) - C(x_1x_2) - Dx_3(\omega x_1 + \omega^2 x_2^2)
\end{array}\right\} . \qquad \text{V.}$$

Of this system of conics three only are real; the remaining six are imaginary.

74. The Hessian transformation of the product of the equations of the two first of the inflectional tangents, Table II., Art. 68, gives the following equation of a conic which has double osculation with the Hessian :—

$$(m^2 - 14m^5 + 4m^8)x_1^2 - m(2 + 4m^3 - 9m^5)(\omega x_2^2 + \omega^2 x_3^2)$$
$$- (1 + 4m^3 - 14m^6)x_2x_3 + m^2(2 - 9m^2 + 4m^3)\omega x_1(x_3 + \omega x_2) = 0. \quad (50)$$

The nine inflectional tangents of Art. 68 give thirty-six combinations, 2 by 2: hence they give, by Hessian transformation, thirty-six conics; and since the Hessian being given, there are three cubics of which it is the Hessian, we have, in all, 108 conics having double osculation with the Hessian.

75. The Hessian transformation of the twelve lines through the nine points of inflection are given in the following Table. The numbers on the right denote the lines of which the conics are the transformations:—

$$\left.\begin{array}{ll}
(123). & x_2x_3 - m^2x_1^2 \\
(456). & x_3x_1 - m^2x_2^2 \\
(789). & x_1x_2 - m^2x_3^2 \\
(147). & (1+2m^2)(x_1x_2+x_2x_3+x_3x_1) - (m^2+2m)(x_1^2+x_2^2+x_3^2) \\
(159). & (1+2\omega m^2)(x_1x_2+\omega^2x_2x_3+x_3x_1) - (m^2+2\omega m)(\omega^2x_1^2+x_2^2+x_3^2) \\
(168). & (1+2\omega^2m^2)(x_1x_2+\omega x_2x_3+x_3x_1) - (m^2+2\omega^2m)(\omega x_1^2+x_2^2+x_3^2) \\
(258). & (1+2m^2)(x_2x_3+\omega^2x_3x_1+\omega x_1x_2) - (m^2+2m)(x_1^2+\omega^2x_2^2+\omega x_3^2) \\
(348). & (1+2\omega m^2)(x_2x_3+\omega^2x_3x_1+x_1x_2) - (m^2+2\omega m)(x_1^2+\omega^2x_2^2+x_3^2) \\
(249). & (1+2\omega^2m^2)(x_2x_3+\omega x_3x_1+x_1x_2) - (m^2+2\omega^2m)(x_1^2+\omega x_2^2+x_3^2) \\
(369). & (1+2m^2)(x_2x_3+\omega x_3x_1+\omega^2x_1x_2) - (m^2+2m)(x_1^2+\omega x_2^2+\omega^2x_3^2) \\
(267). & (1+2\omega m^2)(x_2x_3+\omega^2x_1x_2+x_3x_1) - (m^2+2\omega m)(x_1^2+x_2^2+\omega^2x_3^2) \\
(357). & (1+2\omega^2m^2)(x_2x_3+\omega x_1x_2+x_3x_1) - (m^2+2\omega^2m)(x_1^2+x_2^2+\omega x_3^2)
\end{array}\right\} . \quad \text{VI.}$$

76. The system of conics in the foregoing Table VI. is very remarkable. The following is a statement of some of their properties.

1°.—They can be divided into four sets of three conics each, each set touching the Hessian at the nine points corresponding to the nine points of inflection, namely, the four triads:

(123), (456), (789); (147), (258), (369);
(159), (348), (267); (168), (249), (375);

2°.—The Hessian is touched at each contact by four conics of the system. Thus the conics (123), (147), (159), (168) all touch the Hessian at the point whose co-ordinates are 1, m, m. The equation of the tangent which is common to the Hessian and these four points at this point is $x_2 + x_3 + 2mx_1$. Similar properties hold for the other points of contact.

3°.—Each conic of the system is touched by all the remaining conics except two. Thus denoting the preceding system by the notation σ_1, σ_2, $\sigma_3 \ldots \sigma_{12}$, we find that σ_1 is touched at (1) by the conics σ_4, σ_5, σ_6; at (2) by σ_7, σ_9, σ_{11}; at (3) by σ_8, σ_{10}, σ_{12}; and similarly for the other contacts.

77. If we put, for shortness, the equation of the tangent $x_2 + x_3 - 2mx_1$ of the last Article $= \tau$, we find, after an easy calculation, the three following identities, viz. :—

$$(1 + 4m + 4m^2)\sigma_1 + \{(1 - 2m^2 - 2m^3)x_1 - (m^2 + 2m)(x_2 + x_3)\}\,\tau = \sigma_4,$$

$$(1 + 4\omega m + 2(1+\omega)m^2)\sigma_1 + \{(\omega - 2\omega m^2 - (1+\omega)m^3)x_1 - (m^2 + 2\omega m)(x_2 + x_3)\}\tau = \sigma_5,$$

$$\{1 + 4\omega^2 m + 2(1+\omega^2)m\}\sigma_1 + \{(\omega^2 - 2\omega^2 m^2 - (1+\omega^2)m^3)x_1 - (m^2 + 2\omega^2 m)(x_2 + x_3)\}\tau = \sigma_6.$$

Hence, besides the tangent τ, the other common chords of σ_1, and the conics σ_4, σ_5, σ_6, are the lines

$$(1 - 2m^2 - 3m^3)x_1 - (m^2 + 2m)(x_2 + x_3) = 0, \tag{51}$$

$$\{\omega(1 - 2m^2) - (1 + \omega)m^3\}\,x_1 - (m^2 + 2\omega m)(x_2 + x_3) = 6, \tag{52}$$

$$\{\omega^2(1 - 2m^2) - (1 + \omega^2)m^3\}\,x_1 - (m^2 + 2\omega^2 m)(x_2 + x_3) = 0, \tag{53}$$

and these chords pass through the point of intersection of x_1 with $x_1 + x_2 + x_3$, that is, through the point of inflection on the Hessian to which the point of common contact of the conics corresponds. Hence we have the following remarkable theorem: *The Hessian transformation of any four of the twelve lines passing through a common point of inflection are four conics touching each other, and touching the Hessian at the corresponding point; and the common tangent at that point and the six corresponding common chords of these four conics pass through the original point of inflection.* In other words: *through each point of inflection there passes one line which touches at a common point the four conics which are the Hessian transformations of the four inflectional chords passing through that point, and six lines which are common chords of the same four conics.*

78. The polar conic of a point of inflection consists of two lines, viz., these are the tangents to the Hessian at the corresponding point and the harmonic polar. The following Table contains the Hessian transformations of the nine harmonic polars. Each transformation breaks up into two lines: thus, from the point 1) can be drawn three tangents; one of these is the tangent at the corresponding point on the Hessian, and the others are the Hessian transformation of the harmonic polar:—

$$\left.\begin{array}{ll}
1). & x_1(x_2+x_3) - m^2(x_2+x_3)^2 + 2mx_1^2 \\
2). & x_1(\omega x_2+\omega^2 x_3) - m^2(\omega x_2+\omega^2 x_3)^2 + 2mx_1^2 \\
3). & x_1(\omega^2 x_2+\omega x_3) - m^2(\omega^2 x_2+\omega x_3)^2 + 2mx_1^2 \\
4). & x_2(x_3+x_1) - m^2(x_3+x_1)^2 + 2mx_2^2 \\
5). & x_2(\omega x_3+\omega^2 x_1) - m^2(\omega x_3+\omega^2 x_1)^2 + 2mx_2^2 \\
6). & x_2(\omega^2 x_3+\omega x_1) - m^2(\omega^2 x_3+\omega x_1)^2 + 2mx_2^2 \\
7). & x_3(x_1+x_2) - m^2(x_1+x_2)^2 + 2mx_3^2 \\
8). & x_3(\omega x_1+\omega^2 x_2) - m^2(\omega x_1+\omega^2 x_2)^2 + 2mx_3^2 \\
9). & x_3(\omega^2 x_1+\omega x_2) - m^2(\omega^2 x_1+\omega x_2)^2 + 2mx_3^2
\end{array}\right\}. \qquad \text{VII.}$$

79. The Hessian transformation of the other factor in the polar conic of a point of inflection, viz., the tangent to the Hessian at the point corresponding to the point of inflection, must be a conic having double contact with the Hessian, one contact being at the point of inflection and four pointic; the other contact, at the corresponding point and two pointic: hence this conic must break up into two lines, namely, the tangents to the Hessian at the point of inflection, and its corresponding point. We can easily rectify this theorem by calculation.

Thus the Hessian transformation of

$$x_2 + x_3 - 2mx_1$$

is

$$3m^2(x_2+x_3)^2 + (1-4m^3)(x_1x_2+x_3x_1) - 2m(1+3m^3)x_1^2$$
$$= \{x_2+x_3-2m_1\}\{3m^2(x_2+x_3) + (1+2m^3)x_1\}. \qquad (54)$$

80. The equations in Table VII., Art. 78, give an easy method of finding the equations of the two cubics which are companions to a given fundamental cubic. In other words, a cubic being given, to find the equation of the two other cubics which have the same Hessian as the given cubic.

Let the two companion cubics be

$$x_1^3 + x_2^3 + x_3^3 + 6m'x_1x_2x_3,$$

$$x_1^3 + x_2^3 + x_3^3 + 6m''x_1x_2x_3;$$

then the product of the equations of the tangents at the points which correspond to the point of inflection, with respect to these cubics, will, when equated to the product of the same tangents given in Table VII., enable us to get the values of m' and m''.

Thus

$$(x_2 + x_3 - m'x_1)(x_2 + x_3 - 2m''x_1)$$

$$= (x_2 + x_3)^2 - \frac{1}{m^2}\left\{x_1(x_2 + x_3)\right\} + \frac{2}{m}x_1^2.$$

Hence
$$2m' + 2m'' = \frac{1}{m^2}. \tag{55}$$

$$2m'm'' = -\frac{1}{m}. \tag{56}$$

Hence, solving, we get

$$m' = \frac{1 + \sqrt{1 + 8m^3}}{4m^2}. \tag{57}$$

$$m'' = \frac{1 - \sqrt{1 + 8m^3}}{4m^2}. \tag{58}$$

81. If we eliminate m between the two equations (55), (56), and clear the equations (57), (58) of radicals, we get the three following equations connecting the parameters m, m', m'' of three cubics which have a common Hessian, viz. :—

$$2m'^2m''^2 = m' + m''. \tag{59}$$

$$2m''^2m^2 = m'' + m. \tag{60}$$

$$2m^2m'^2 = m + m'. \tag{61}$$

82. Before concluding this section, we give the following theorem on Hessian transformation. It was inadvertently omitted in Section I. of the Chapter to which it properly belongs: "*If two of three conjugate points with respect to a given cubic be consecutive points on a given line, then the locus of the third point will be the Hessian transformation of the given line.*"

CHAPTER III.

SECTION I.—CENTRAL TRANSFORMATION.

83. We have seen, Art. 22, that if the third point, z, of a conjugate system with respect to a given cubic be situated on the line at infinity, then the two remaining points, x and y, will be conjugate centres, that is, if x be the centre of the polar conic of y, y will be the centre of the polar conic of x: *Hence, it follows that there is a* 1-*to*-1 *correspondence between the positions of x and y. Hence, if the point x describe any curve F, the point y will describe a corresponding curve F', which I shall call the central transformation of the curve F.* This transformation, which will lead to some important results, will be the subject of this section. We could generalize it by supposing the third point to move on any finite line N; then x and y would be conjugate poles, that is, if x be the pole of N with respect to the polar conic of y, then y will be the pole of N with respect to the polar conic of x; but this is evidently only the projection of the former theorem: hence, unless the contrary be stated, we shall, for the sake of conciseness of enunciation, suppose the line on which the third point moves to be at infinity.

84. If A_1, A_2, A_3 be the angles of the triangle of reference, the equation of the line at infinity is $x_1 \sin A_1 + x_2 \sin A_2 + x_3 \sin A_3 = 0$; or, denoting length of the sides of the triangle by a_1, a_2, a_3, the line at infinity will be $a_1x_1 + a_2x_2 + a_3x_3 = 0$. The Hessian transformation of this line will be the

locus of points whose polar conics are parabolas. We shall denote it for shortness by π, and its differentials with respect to a_1, a_2, a_3 by π_1, π_2, π_3 respectively.

85. If x_1, x_2, x_3 be the co-ordinates of any point x, we find, without difficulty, that the co-ordinates of the centre of the polar conic of x are the results of substituting x_1, x_2, x_3 in π_1, π_2, π_3; in other words, the co-ordinates of the centre conjugate to x (see Art. 83) will be the result of that substitution. Hence, if the point x describe any curve

$$f(x_1,\ x_2,\ x_3) = 0,$$

its conjugate centre will describe the curve

$$f(\pi_1,\ \pi_2,\ \pi_3) = 0,$$

and we have therefore the following theorem:

The central transformation of the curve

$$f(x_1,\ x_2,\ x_3) = 0$$

is

$$f(\pi_1,\ \pi_2,\ \pi_3) = 0. \tag{62}$$

86. The equation of the three conics π_1, π_2, π_3, written in full, are

$$\begin{aligned}\pi_1 &\equiv a_1A_{11} + a_2A_{12} + a_3A_{13},\\ \pi_2 &\equiv a_1A_{21} + a_2A_{22} + a_3A_{23},\\ \pi_3 &\equiv a_1A_{13} + a_2A_{32} + a_3A_{33},\end{aligned}$$

and attending to the umbral signification of A_1, A_2, A_3 (see Art. 31), these may be written

$$\begin{aligned}\pi_1 &\equiv A_1(a_1A_1 + a_2A_2 + a_3A_3),\\ \pi_2 &\equiv A_2(a_1A_1 + a_2A_2 + a_3A_3),\\ \pi_3 &\equiv A_3(a_1A_1 + a_2A_2 + a_3A_3).\end{aligned}$$

Hence, for points on the Hessian, for which A_1, A_2, A_3 have interpretable significations, π_1, π_2, π_3 are proportional to A_1, A_2, A_3: hence we have the following theorem: *The central and Hessian transformations of any curve meet the Hessian of the fundamental cubic in the same point.*

87. If the third point z move on a finite line,

$$N = l_1 z_1 + l_2 z_2 + l_3 z_3,$$

then we find, as before,

$$\pi_1 = A_1(l_1 A_1 + l_2 A_2 + l_3 A_3),$$
$$\pi_2 = A_2(l_1 A_1 + l_2 A_2 + l_3 A_3),$$
$$\pi_3 = A_3(l_1 A_1 + l_2 A_2 + l_3 A_3);$$

and these for points on the Hessian are proportional to A_1, A_2, A_3, and therefore independent of l_1, l_2, l_3; or, in other words, the co-ordinates of the point which corresponds to any point on the Hessian are independent of the position of the line N, on which the third point of the three conjugate points moves.

Cor.—In the theorems of the two last Articles, if the fundamental curve be not a cubic, we must substitute the Steinerian for the Hessian.

88. We infer, from Art. 38, *Cor.* 2, that the three conics, π_1, π_2, π_3 have common the three points on the Hessian which correspond to the points at infinity on the Hessian: hence (see Salmon's *Higher Curves*, Art. 353), *our central transformation* is a "*Cremona transformation,*" *the principal points of which are the points which correspond to the points at infinity on the Hessian.*

89. Since the central transformation is a 1-to-1 transformation, we see that if two curves have any contacts, the transformed curves will have corresponding contacts. Thus, if two conics have double contact, the quartics into which they transform will have double contact. Again, since the curves π_1, π_2, π_3 have three points common, the transformed of any curve of degree k will be a curve of degree $2k$, having the three principal points as multiple points of the order k. Lastly, the transformed of any unicursal curve will be an unicursal curve; thus, the transformed of a conic will be an universal quartic. The multiple points in the transformed of any curve will be the points which correspond to the multiple points in the original curve, together with the principal points.

90. The central transformation of the line

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$

is the conic

$$\lambda_1 \pi_1 + \lambda_2 \pi_2 + \lambda_3 \pi_3 = 0;$$

or, say

$$\lambda_\pi = 0.$$

Hence, if a variable point move along the line λ_x, the centre conjugate to it moves along the conic $\lambda_\pi = 0$; and, since this property is reciprocal, it follows that if a variable point move along the conic λ_π, the centre conjugate to it will move along the line λ_x. Now, the central transformation of a conic is a quartic; hence the central transformation of λ_π breaks up into the line λ_x and a cubic, say Ω. The curve Ω (see Salmon's *Higher Curves*, page 308, Art. 360) is the Jacobian of the three conics π_1, π_2, π_3; but when three conics have three points common, their Jacobian, which must have these as double points, consists of the three lines joining them. Hence we have the following theorem: *The central transformation of any conic passing through the three principal points breaks up into four lines, namely, the three lines joining these points, and a fourth line, of which the conic is the central transformation.*

91. We shall denote the equation of the three lines through the principal points, as in the last Article, by Ω: its equation will be an important one in our investigation. The theorem of the last Article may be thus stated: The second central transformation of any line L (that is, the transformation of the transformation) is equal to ΩL; hence we have the following special cases: the second transformations of x_1, x_2, x_3 are Ωx_1, Ωx_2, Ωx_3 respectively.

92. To find the central transformation of x_1^2, this is

$$\pi_1^2, \text{ or } (a_1 A_{11} + a_2 A_{12} + a_3 A_{13})^2.$$

Squaring out, we find the

$$\text{coefficient of } a_1^2 = A_{11}\,A_{11},$$

$$\text{,,} \qquad a_2^2 = A_{11}\,A_{22} - Hx_3,$$

$$\text{,,} \qquad a_3^2 = A_{11}\,A_{33} - Hx_2,$$

$$\text{,,} \qquad a_2a_3 = 2A_{11}\,A_{23} + 2mHx_1,$$

$$\text{,,} \qquad a_3a_1 = 2A_{11}\,A_{31},$$

$$\text{,,} \qquad a_1a_2 = 2A_{11}\,A_{12}:$$

hence the required transformation is

$$A_{11}\pi - H(a_2^2x_3 + a_3^2x_2 - 2ma_2a_3x_1), \tag{63}$$

where H denotes the Hessian.

Cor.—In like manner, the transformation

of $$x_2^2 = A_{22}\pi - H(a_3^2x_1 + a_1^2x_3 - 2ma_3a_1x_2); \tag{64}$$

,, $$x_3^2 = A_{33}\pi - H(a_1^2x_2 + a_2^2x_1 - 2ma_1a_2x_3). \tag{65}$$

93. The transformation of x_1x_2 is $\pi_1\pi_2$, or

$$(a_1A_{11} + a_2A_{12} + a_3A_{13})(a_1A_{21} + a_2A_{22} + a_3A_{23}):$$

hence, multiplying out, and calculating as in the last Article, we find the required transformation to be

$$A_{12}\pi - H(2ma_3a_1x_1 + 2ma_3a_2x_2 - 2a_1a_2x_3 - ma_3^2x_3). \tag{66}$$

In the same manner we find the transformation

of $$x_2x_3 = A_{23}\pi - H(2ma_1a_2x_2 + 2ma_1a_3x_3 - 2a_2a_3x_1 - ma_1^2x_1); \tag{67}$$

,, $$x_3x_1 = A_{31}\pi - H(2ma_2a_3x_3 + 2ma_2a_1x_1 - 2a_3a_1x_2 - ma_2^2x_2). \tag{68}$$

94. From the results of the two last Articles we have at once the central transformation of the conic $(a, b, c, f, g, h)(x_1, x_2, x_3)^2$; viz., this is equal to

$$H_1\pi + HL, \tag{69}$$

where H_1 denotes the Hessian transformation of the conic (see Art. 31), and L is the equation of the line which is the locus of points, whose polar conics cut the line at infinity in points which are harmonic conjugates to the points in which infinity is intersected by the conic to be transformed.

95. We are now able to find the value of Ω (*see* Art. 91).

Since the second transformation of the line x is Ωx, it follows that Ωx is the first transformation of the conic π_1. Now, writing π_1 at length, its equation is

$$a_1A_{11} + a_2A_{12} + a_3A_{13} = 0;$$

or,

$$a_1(x_2x_3 - m^2x_1^2) + a_2(m^2x_1x_2 - mx_3^2) + a_3(m^2x_1x_3 - m^2x_2^2) = 0.$$

Hence, by the last Article, the central transformation of this is

$$\{KH - S(a_1x_1 + a_2x_2 + a_3x_3)\pi\}x, \tag{70}$$

where K is put for shortness for the expression

$$m(a_1^3 + a_2^3 + a_3^3) + (1 - 4m^3)a_1a_2a_3;$$

hence, equating the equation (60) with Ωx, we have

$$\Omega = KH - (m - m^4)(a_x)\pi:$$

hence, *the equation of the three lines joining the points on the Hessian corresponding to the three points at infinity is*

$$KH - S\pi a_x = 0. \tag{71}$$

Cor.—If in the equation (61) we replace a_1, a_3, a_3 by λ_1, λ_2, λ_3, we get

the equation of the lines joining the points corresponding to the three points where the line λ_x meets the Hessian, viz.,

$$KH - SE\lambda_x = 0\,; \tag{72}$$

where K denotes the Cayleyan

$$m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3,$$

E the Hessian transformation of λ_x, and S the invariant $m - m^4$.

SECTION II.—SPECIAL TRANSFORMATIONS.

96. Find the central transformation of $\frac{dU}{dx_1}$ where U is the fundamental cubic,

$$\frac{dU}{dx_1} = x_1^2 + 2mx_2x_3:$$

hence the transformation is

$$\pi_1^2 + 2m\pi_2\pi_3\,;$$

and substituting for π_1, π_2, π_3 their values from Article 86, we find as follows:—

$$\text{coefficient of } a_1^2 = A_{11}\,.\,\frac{dH}{dx_1} + 2m^2Hx_1,$$

$$\text{,,} \quad a_2^2 = A_{22}\,.\,\frac{dH}{dx_1} - Hx_3,$$

$$\text{,,} \quad a_3^2 = A_{33}\,.\,\frac{dH}{dx_1} - Hx_2,$$

$$\text{,,} \quad a_2a_3 = 2A_{23}.\frac{dH}{dx_1} + 4mHx_1,$$

$$\text{,,} \quad a_3a_1 = 2A_{31}.\frac{dH}{dx_1} - 2m^2Hx_3,$$

$$\text{,,} \quad a_1a_2 = 2A_{12}.\frac{dH}{dx_1} + 2m^2Hx_2:$$

hence, collecting terms, we find the central transformation of $\frac{dU}{dx_1}$ to be equal

$$\pi\frac{dH}{dx_1} - H\frac{d\pi}{dx_1}. \tag{73}$$

Cor.—The central transformation of the polar conic of any point with respect to the fundamental cubic is

$$\pi H' - H\pi' = 0, \tag{74}$$

where H' and π' are the polar conic and the polar line of the same point with respect to H and π respectively.

97. The central transformation of $\frac{dH}{dx_1}$ will be found in the same manner to be

$$3H\frac{dQ}{dx_1} - S\pi\frac{dU}{dx_1}. \tag{75}$$

When Q denotes the conic,

$$\begin{aligned} m(1 + 2m^3)&(a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 - 2a_2a_3x_2x_3 - 2a_3a_1x_3x_1 - 2a_1a_2x_1x_2) \\ &+ (1 - 4m^3)(a_2a_3x_1^2 + a_3a_1x_2^2 + a_1a_2x_3^2) \\ &+ 6m^2(a_1^2x_2x_3 + a_2^2x_3x_1 + a_3^2x_1x_2), \end{aligned} \tag{76}$$

or *the locus of points whose polar conics with respect to the cubic and Hessian are cut harmonically by the line at infinity.*

98. *The central transformation of the Hessian is ΩH.*

Demonstration.—The coefficient of a_1^3 in ΩH is easily seen to be

$$H\{3m^4x_1x_2x_3 - m^6x_1^3 - m^3x_2^3 - m^3x_3^3\},$$

and the coefficient of a_1^3, in the central transformation of H, is

$$\begin{aligned} &- m^2\{(x_2x_3 - m^2x_1^2)^3 + (m^2x_1x_2 - mx_3^2)^3 + (m^2x_3x_1 - mx_2^2)^3\} \\ &+ (1 - 2m^3)(x_2x_3 - m^2x_1^2)(x_3x_1 - m^2x_2^2)(x_1x_2 - m^2x_3^2), \end{aligned}$$

and these are equal. Similarly for the other coefficients. Hence the proposition is proved.

99. Since $2\pi = a_1\pi_1 + a_2\pi_2 + a_3\pi_3$, we have the transformation of 2π equal

$$\Omega\,(a_1x_1 + a_2x_2 + a_3x_3,$$

and the transformation of $a_1x_1 + a_2x_2 + a_3x_3$ is 2π: hence, the transformation of $\pi(a_1x_1 + a_2x_2 + a_3x_3)$ is

$$\Omega\pi(a_1x_1 + a_2x_2 + a_3x_3). \tag{77}$$

100. From the two last Articles we see that the transformation of $KH - S\pi(a_1x_1 + a_2x_2 + a_3x_3)$ is

$$\Omega(KH - S\pi a_x) \text{ or } \Omega^2:$$

that is, the central transformation of Ω is Ω^2. Hence we have the following important theorem: *The centre of the polar conic of any point on the three lines represented by Ω lies on the same three lines.* We shall get an independent proof of this theorem in the next section.

101. If $U = 0$ be the fundamental cubic, we have seen, Art. 5, that the Hessian of the cubic

$$\alpha U + \beta(\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3)^3 \text{ is } \alpha H + \beta E\lambda_x:$$

hence, conversely, we infer that the cubic Ω is the Hessian of the cubic

$$KU - S(a_1x_1 + a_2x_2 + a_3x_3)^3:$$

hence we can apply to Ω all the theorems of the Hessian, for doing which the following principle will be useful:—

The polar conics of any point with respect to two cubics, which have the line at infinity as a chord of osculation, have the same asymptotes. This follows from Art. 15.

102. The envelope of the asymptotes of the polar conics of all the points on Ω is a curve of the third class. This is evident from the last Article—in fact the envelope is the Cayleyan of the cubic

$$KU - S(a_1x_1 + a_2x_2 + a_3x_3)^3.$$

Its equation can be inferred from Art. 52; it is the discriminant of the

Hessian transformation of the product of λ_x and the line at infinity, and we see, from Art. 53, that this contravariant represents the three angular points of the triangle represented by Ω.

103. Since Ω is the product of three lines, and since it is the Hessian of the cubic $KU - S(a_1x_1 + a_2x_2 + a_3x_3)^3$, it follows that this cubic is the sum of the cubes of the three lines denoted by Ω; therefore U is the sum of four cubes. Hence we have the following important theorem: *Every cubic can be represented as the sum of the cubes of four lines, one of which, L, may be taken arbitrarily; the other three will then be determined as the sides of the triangle formed by joining the points on the Hessian of the cubic corresponding to the points where the line L intersects it.*

We shall call the complete quadrilateral formed by the four lines in this theorem a *self-conjugate quadrilateral.*

104. If y_1, y_2, y_3 be the co-ordinates of any point, then the co-ordinates of its corresponding point will be the result of substituting these co-ordinates in π_1, π_2, π_3. Again, the polar line of the second point is

$$x_1\left(\frac{dU}{dx_1}\right) + x_2\left(\frac{dU}{dx_2}\right) + x_3\left(\frac{dU}{dx_3}\right),$$

where π_1, π_2, π_3 are substituted for the variables in the differentials; or, in other words, where the differentials are to be transformed by the central transformation. Hence, the equation of the polar line of the point corresponding to the point (y) is (see Art. 96)

$$x_1\left(\pi\frac{dH}{dx_1} - H\frac{d\pi}{dx_1}\right) + x_2\left(\pi\frac{dH}{dx_2} - H\frac{d\pi}{dx_2}\right) + x_3\left(\pi\frac{dH}{dx_3} - H\frac{d\pi}{dx_3}\right) = 0, \quad (78)$$

where y_1, y_2, y_3 are to be substituted for x_1, x_2, x_3 in the quantities within the brackets. Hence we have the following theorem: *If A and B be two corresponding points, then the polar lines of A with respect to H and π, and the polar line of B with respect to U, are concurrent.* This is a generalization of known theorems, as will be seen from the following corollaries:—

Cor. 1.—If the point A lies on H, the point B also lies on it; and the result of substituting the co-ordinates of B, in equation (78), becomes

$$x_1\left(\frac{dH}{dx_1}\right) + x_2\left(\frac{dH}{dx_2}\right) + x_3\left(\frac{dH}{dx_3}\right) = 0,$$

where y_1, y_2, y_3 are to be substituted in the differentials: *Hence, if A and B be corresponding points on the Hessian, the polar line of A with respect to U is the tangent at B to the Hessian.*

Cor. 2.—*If the point A be at infinity, the point B lies on π: hence, the polar line of A with respect to U is the tangent at B to π.*

105. From equation (75) we can, in exactly the same way, infer the following theorem: *If A and B be any two corresponding points, then the polar lines of the point A with respect to Q and U, and the polar line of B with respect to H, are concurrent.*

106. By combining the theorems of the two last propositions, we have the following proposition: *If A and B be any two conjugate centres, then are concurrent the four lines, namely, the polar lines of A with respect to Q and U, and the polar lines of B with respect to π and H.*

107. Since the equation of a pair of circles touching three given circles is of the form

$$l\sqrt{S_1} + m\sqrt{S_2} + n\sqrt{S_3} = 0,$$

and since the central transformation of a conic is a quartic having the principal points as double points, we have the following theorem: *If σ_1, σ_2, σ_3 be three unicursal quartics having the same three double points, and also two other single points common, then the equation of a pair of quartics having the same double and common points, and touching the quartics σ_1, σ_2, σ_3, will be of the form*

$$l\sqrt{\sigma_1} + m\sqrt{\sigma_2} + n\sqrt{\sigma_3} = 0. \tag{79}$$

108. If $S^{\frac{1}{2}} - L$, $S^{\frac{1}{2}} - M$, $S^{\frac{1}{2}} - N$ be three conics having double contact with a given conic S, the equation of a pair of conics touching them is of the form

$$\sqrt{\lambda(S^{\frac{1}{2}} - L)} + \sqrt{\mu(S^{\frac{1}{2}} - M)} + \sqrt{\nu(S^{\frac{1}{2}} - N)} = 0;$$

and there are sixteen such pairs (see *Bicircular Quartics*, page 70). Hence, by Central transformation, we have the following theorem: *If $S - L^2$, $S - M^2$, $S - N^2$ be three unicursal quartics having common nodes, and L, M, N three conics passing through the nodes, then the equation of a pair of quartics, having, besides the same common nodes, double contact with S and touching the quartics $S - L^2$, $S - M^2$, $S - N^2$, is of the form*

$$\sqrt{\lambda(S^{\frac{1}{2}} - M)} + \sqrt{\mu(S^{\frac{1}{2}} - M)} + \sqrt{\nu(S^{\frac{1}{2}} - N)} = 0; \qquad (80)$$

and sixteen such pairs of quartics can be described.

109. We conclude this section by the solution of a few problems.

1°. To find the condition that the central transformation of the equation of a given line may touch the line:—

Let the line be λ_x, then the transformation is λ_π, and the discriminant of this is the determinant—

$$\begin{vmatrix} 2\{m^2a_1\lambda_1 + m(a_2\lambda_3 + a_3\lambda_2)\} & -a_3\lambda_3 + m^2(a_1\lambda_2 + a_2\lambda_1\} & -\{a_2\lambda_2 + m^2(a_3\lambda_1 + a_1\lambda_3)\} \\ -\{a_3\lambda_3 + m^2(a_1\lambda_2 + a_2\lambda_1)\} & 2\{m^2a_2\lambda_2 + m(a_3\lambda_1 + a_1\lambda_3)\} & -\{a_1\lambda_1 + m^2(a_2\lambda_3 + a_3\lambda_2)\} \\ -\{a_2\lambda_2 + m^2(a_3\lambda_1 + a_1\lambda_3)\} & -\{a_1\lambda_1 + m^2(a_2\lambda_3 + a_3\lambda_2)\} & 2\{m^2a_3\lambda_3 + m(a_1\lambda_2 + a_2\lambda_1)\} \end{vmatrix}.$$

Then the required condition is this determinant bordered with the line co-ordinates λ_1, λ_2, λ_3, and this, as is easily seen, will be the tangential equation of the four poles of the line at infinity.

Cor.—If in the result of this Article we replace the co-ordinates a_1, a_2, a_3 of the line at infinity by the co-ordinates of a finite line μ_x, the result will be the tangential equation of the poles of μ_x.

2°. Find the locus of a point when the central transformation of its polar line, with respect to the conic π, passes through the point itself:—

If the co-ordinates of the point $\alpha_1, \alpha_2, \alpha_3$, its polar line, with respect to π, will be

$$\alpha_1\frac{d\pi}{dx_1} + \alpha_2\frac{d\pi}{dx_2} + \alpha_3\frac{d\pi}{dx_3} = 0.$$

Hence the conditions of the question give us, after replacing $\alpha_1, \alpha_2, \alpha_3$ by x_1, x_2, x_3,

$$\frac{d\pi}{da_1}\cdot\frac{d\pi}{dx_1} + \frac{d\pi}{da_2}\cdot\frac{d\pi}{dx_2} + \frac{d\pi}{da_3}\cdot\frac{d\pi}{dx_3} = 0. \tag{81}$$

Cor.—This equation is easily seen to be equal

$$Qa_x - KH = 0; \tag{82}$$

where Q denotes the conic equation (76), Art. 97, and K the result of substituting a_1, a_2, a_3 for the line co-ordinates in the Cayleyan.

3°. Find the condition that the central transformation of the polar line of a point, with respect to the cubic, may pass through the point:—

This will evidently be

$$\pi_1 U_1 + \pi_2 U_2 + \pi_3 U_3 = 0;$$

or the determinant,

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & U_1 \\ u_{21}, & u_{22}, & u_{23}, & U_2 \\ u_{31}, & u_{32}, & u_{33}, & U_3 \\ a_1, & a_2, & a_3 & \end{vmatrix} = 0;$$

and this is evidently

$$H(a_1x_1 + a_2x_2 + a_3x_3) = 0. \tag{83}$$

4°. The locus of a point, whose polar line with respect to the Hessian will be a diameter of its polar conic with respect to the cubic, may be found from the consideration that, since the diameter of the polar conic passes through the centre of the polar conic, its central transformation passes through the original point.

Now, if the point be (α), its polar line with respect to the Hessian will be

$$x_1\left(\frac{dH}{dx_1}\right) + x_2\left(\frac{dH}{dx_2}\right) + x_3\left(\frac{dH}{dx_3}\right) = 0,$$

where the co-ordinates of α are substituted for x_1, x_2, x_3 in the differentials. Hence, the conditions of the question give us for the required locus

$$\pi_1 \cdot \left(\frac{dH}{dx_1}\right) + \pi_2 \cdot \left(\frac{dH}{dx_2}\right) + \pi_3 \cdot \left(\frac{dH}{dx_3}\right) = 0;$$

or, as it may be written,

$$\pi_H = 0. \qquad (84)$$

5°. To find the central transformation of U, that is, to find the locus of the centre of the polar conics of all the points on U:—

Since

$$3U = x_1 \frac{dU}{dx_1} + x_2 \frac{dU}{dx_2} + x_3 \frac{dU}{dx_3},$$

we have, by Art. 96, the required transformation given by the equation

$$\pi_1\left(\pi \cdot \frac{dH}{dx_1} - H \cdot \frac{d\pi}{dx_1}\right) + \pi_2\left(\pi \cdot \frac{dH}{dx_2} - H \cdot \frac{d\pi}{dx_2}\right) + \pi_3\left(\pi \cdot \frac{dH}{dx_3} - H \cdot \frac{d\pi}{dx_3}\right);$$

or,

$$\pi\left(\pi_1 \cdot \frac{dH}{dx_1} + \pi_2 \frac{dH}{dx_2} + \pi_3 \frac{dH}{dx_3}\right)$$

$$- H\left(\pi_1 \frac{d\pi}{dx_1} + \pi_2 \frac{d\pi}{dx_2} + \pi_3 \frac{d\pi}{dx_3}\right) = 0. \qquad (85)$$

The value of the second bracket is given in equation (82), and the first bracket is equal to the symbolical product

$$\left(a_1A_1 + a_2A_2 + a_3A_3\right)\left(A_1 \frac{dH}{dx_1} + A_2 \frac{dH}{dx_2} + A_3 \frac{dH}{dx_3}\right);$$

or,

$$\left(A_a\right)\left(A_{\frac{dH}{dx}}\right). \qquad (86)$$

110. *If in the central transformation we join corresponding points or conjugate centres, the envelope of the joining line will be a curve whose class is three times the degree of the curve to be transformed.* For, by a known principle, if there is a 1-to-1 correspondence between the points on two curves, the class of the envelope of the lines joining corresponding points will be equal to the sum of the degrees of the two curves. Now, if the curves be f and F, the degree of the latter is twice that of the former: hence the proposition is proved.

Cor.—If f be a line, the envelope will have f as a double tangent. For, let R and S be the points where the transformed of f cuts f, and let P and Q be the points on f which correspond to R and S. Now, f is a tangent to the envelope, because it joins the points P and R; and it is a tangent because it joins Q and S: hence it is a double tangent.

Section III.—Self-Conjugate Tetragrams.

111. We have seen, Art. 103, that the equation of every cubic can be written in the form

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^4 = 0, \tag{87}$$

where a_1, a_2, a_3, a_4 are four parameters, and $x_1 + x_2 + x_3 + x_4 = 0$. We shall devote this section to this form of equation; and we shall thus arrive at independent proofs of some of our previous results, and also at some new ones of an interesting nature. The Hessian of the cubic

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^4$$

is

$$\begin{vmatrix} a_1x_1 + a_4x_4, & a_4x_4, & a_4x_4 \\ a_4x_4, & a_2x_2 + a_4x_4, & a_4x_4 \\ a_4x_4, & a_4x_4, & a_3x_3 + a_4x_4 \end{vmatrix} = 0. \tag{88}$$

Hence, expanding and dividing by $a_1a_2a_3a_4x_1x_2x_3x_4$, we get

$$(a_1x_1)^{-1} + (a_2x_2)^{-2} + (a_3x_3)^{-1} + (a_4x_4)^{-1} = 0. \tag{89}$$

112. The Hessian transformation of the line $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$, with respect to the cubic, may be found thus: writing the equation in the form $(\lambda_1 - \lambda_4)x_1 + (\lambda_2 - \lambda_4)x_2 + (\lambda_3 - \lambda_4)x_3$, the required transformation will be got by bordering the determinant (88) with $\lambda_1 - \lambda_4$, $\lambda_2 - \lambda_4$, $\lambda_3 - \lambda_4$; and expanding, we get, after an easy reduction,

$$(\lambda_1 - \lambda_2)^2(a_3 a_4 x_3 x_4) + (\lambda_1 - \lambda_3)^2(a_2 a_4 x_2 x_4) + (\lambda_1 - \lambda_4)^2(a_2 a_3 x_2 x_3)$$
$$+ (\lambda_2 - \lambda_3)^2(a_1 a_4 x_1 x_4) + (\lambda_2 - \lambda_4)^2(a_1 a_3 x_1 x_3) + (\lambda_3 - \lambda_4)^2(a_1 a_2 x_1 x_2) = 0. \tag{90}$$

Cor.—The Hessian transformations of the four fundamental lines, x_1, x_2, x_3, x_4, are

$$(a_2 x_2)^{-1} + (a_3 x_3)^{-1} + (a_4 x_4)^{-1} = 0; \tag{91}$$
$$(a_3 x_3)^{-1} + (a_4 x_4)^{-1} + (a_1 x_1)^{-1} = 0; \tag{92}$$
$$(a_4 x_4)^{-1} + (a_1 x_1)^{-1} + (a_2 x_2)^{-1} = 0; \tag{93}$$
$$(a_1 x_1)^{-1} + (a_2 x_2)^{-1} + (a_3 x_3)^{-1} = 0:$$

hence, the Hessian transformation of each of the fundamental lines is a conic described about the triangle formed by the three remaining lines, and touching the Hessian at the angular points of the same triangle.

113. The three pairs of tangents to the Hessian at the three pairs of corresponding points are

$$\begin{array}{ll} a_1 x_1 + a_2 x_2, & a_3 x_3 + a_4 x_4, \\ a_2 x_2 + a_3 x_3, & a_4 x_4 + a_1 x_1, \\ a_3 x_3 + a_1 x_1, & a_2 x_2 + a_4 x_4; \end{array}$$

and these three pairs of tangents meet respectively on the Hessian in three points which are collinear, namely, on the line

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0. \tag{94}$$

This line may be called the axis of the tetragrams (*see* "On the Equation of Circles" (Second Memoir), *Transactions of the Royal Irish Academy*, vol. xxvi.).

114. *The axis of the tetragram is the satellite of each of the four sides of the tetragram with respect to the Hessian.* This follows from the fact that the six tangents at the points (x_1x_2), (x_3x_4), &c., meet the Hessian on that line.

115. The three points on the Hessian, which correspond to the points where the axis meets it, are the three points of intersection of the three lines of connexion of corresponding points, or, in other words, the three diagonals of the quadrilateral formed by the four fundamental lines. The equation of the conic touching the Hessian in these points is

$$\frac{(a_1-a_2)^2}{a_1a_2x_1x_2}+\frac{(a_1-a_3)^2}{a_1a_3x_1x_3}+\frac{(a_1-a_4)^2}{a_1a_4x_1x_4}$$
$$+\frac{(a_2-a_3)^2}{a_2a_3x_2x_3}+\frac{(a_3-a_4)^2}{a_3a_4x_3x_4}+\frac{(a_4-a_2)^2}{a_4a_2x_4x_2}=0. \tag{95}$$

116. *If a variable point P moves along any of the four fundamental lines, the triangle formed by the three remaining lines will be self-conjugate with respect to the polar conic of P.*

Demonstration.—The polar conic of the point (y), with respect to the cubic, will be

$$a_1y_1x_1^2+a_2y_2x_2^2+a_3y_3x_3^2+a_4y_4x_4^2=0.$$

Hence, making one of the y's such as $y_4=0$, the proposition is proved.

119. If the triangle formed by the three lines Ω be denoted by LMN, then, by the last article, we have the following theorem: *The polar conics of all the points on any side MN of the triangle LMN are concentric, their common centre being the opposite angular point L of the triangle.*

Cor.—Hence we have independent proof of the theorem of Art. 102, namely, that the Cayleyan breaks up into three points.

118. Since the polar conic of any point on x_3 is of the form

$$a_1y_1x_1^2 + a_2y_2x_2^2 + a_4y_4x_4^2 = 0;$$

hence, if $x_4 = 0$ be the line at infinity, the asymptotes of the polar conic of any point on x_3 will be of the form

$$a_1y_1x_1^2 + a_2y_2x_2^2 = 0.$$

If these be real asymptotes, a_1y_1 and a_2y_2 will have contrary signs; and, putting $a_2y_2 \div a_1y_2 = -k^2$, the equation of the asymptotes will be

$$x_1^2 - k^2x_2^2 = 0. \tag{96}$$

Hence we have the following theorem: *If LMN be the triangle formed by joining the points on the Hessian, which correspond to the three points at infinity, then the asymptotes of the polar conics of all the points on any of its sides, such as MN, will form a pencil in involution, the double lines of which will be the remaining sides of the triangle.*

119. The Cayleyan of the cubic $a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3$ is

$$\begin{aligned} &a_2a_3a_4\lambda_1^3 + a_3a_4a_1\lambda_2^3 + a_4a_1a_2\lambda_3^3 \\ &+ (a_2a_3a_4 + a_3a_4a_1 + a_4a_1a_2)\lambda_1\lambda_2\lambda_3 \\ &- (a_2a_3a_4)(\lambda_2+\lambda_3)\lambda_1^2 - (a_3a_4a_1)(\lambda_3+\lambda_1)\lambda_2^2 \\ &- (a_4a_1a_2)(\lambda_1+\lambda_2)\lambda_3^2 - a_1a_2a_3\lambda_1\lambda_2\lambda_3 = 0. \end{aligned}$$

This equation can be written in a symmetrical form by changing λ_1 into $\lambda_1 - \lambda_4$, &c., and the equation will be transformed into

$$\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{a_1} + \frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_2-\lambda_1)}{a_2} + \frac{(\lambda_3-\lambda_4)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{a_3} + \frac{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}{a_4} = 0. \tag{97}$$

Now, if we denote the biquadratic

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4) = 0 \text{ by } f(\lambda),$$

the equation (97) can be written in the form

$$\frac{f'(\lambda_1)}{a_1} + \frac{f'(\lambda_2)}{a_2} + \frac{f'(\lambda_3)}{a_3} + \frac{f'(\lambda_4)}{a_4} = 0. \tag{98}$$

CHAPTER IV.

SECTION. I.—THE QUARTIC.

120. We shall now consider the case mentioned in Art. 30, namely: "If one of the three points of a conjugate system moves along a given conic, and the other two coincide, find the equation of the curve they describe."

This problem is equivalent to the following: If a variable point move along a given conic, find the envelope of its polar conic with respect to a given cubic.

If $U = 0$ be the cubic, and $(a, b, c, d, e, f, g, h)(x_1, x_2, x_3)^2$ the conic, then denoting differentials of U by U_1, U_2, U_3, the required envelope will be

$$\begin{vmatrix} a, & h, & g, & U_1 \\ h, & b, & f, & U_2 \\ g, & f, & c, & U_3 \\ U_1, & U_2, & U_3 & \end{vmatrix} = 0. \tag{99}$$

Now, since the conic contains five constants, and the cubic nine, the general equation just written contains fourteen constants, and is therefore general enough to denote any curve of the fourth degree. Again, if the given cubic has a double point, the differentials U_1, U_2, U_3 will each pass through it. Hence the double point on the cubic will be a double point on the quartic. Similarly, a cusp on the cubic will be a cusp on the quartic.

121. Since the Hessian transformation of U_1 is H_1 (see Art. 42), we have the following theorem: *The two quartics which are the envelopes of the polar conics, with respect to a cubic and its Hessian of a variable point which moves along a given conic, are such that either is the Hessian transformation of the other.* And since, if a cubic has a double point its Hessian has a double point, it follows *that if a quartic has a double point its Hessian transformation has a double point.*

122. The problem solved in Art. 120 is plainly the same as finding the locus of a point whose polar line with respect to the cubic touches a given conic. Let us denote the quartic (99) by Z. We see that the polar line of every point on Z is a tangent to the dirigent conic. But Z intersects U in twelve points; hence, the tangents to U at these points are tangents to the dirigent conic: *Hence, the points of contact on the cubic of common tangents to the cubic and dirigent conic are the points of intersection of the quartic and cubic.*

Cor. 1.—*If the cubic and dirigent conic touch in a point P, then the quartic touches both curves in the same point.*

Cor. 2.—*If the contact of the cubic and conic be three-pointic, four-pointic,* &c., *the contact of the quartic and cubic will be of the same order.*

123. Since Z intersects the Hessian of U in twelve points, and the polar lines of these points are tangents to the Hessian at the corresponding points; hence we have the following theorem: *The twelve points of intersection of the quartic and Hessian are poles of the twelve common tangents of the Hessian and the dirigent conic.*

Cor.—If the Hessian touch the dirigent conic, two common tangents of Hessian and dirigent conic will coincide; and therefore two points of intersection of quartic and Hessian coincide; in other words, the quartic touches the Hessian.

124. The quartic meets the Cayleyan in twenty-four points, and the polar line of each of these points is a tangent to the Hessian, but the Hessian and dirigent conic can have only twelve common tangents. Hence the twenty-four points of intersection of the quartic and Cayleyan must consist of twelve pairs of points, each pair being the points of contact of a pair of lines forming a polar conic of the cubic. We have, therefore, the following construction for these points:—*Find the twelve points on the Hessian which correspond to the points of contact on the Hessian of commom tangents to the Hessian and dirigent conic, and from these points draw to the Cayleyan twelve pairs of corresponding tangents: the points of contact of the latter will be the points required.*

125. We find the tangential equation of the quartic Z as follows. From the definitions it is plain that the Hessian transformation of every tangent line to Z touches the dirigent conic. Hence, if λ_x be a tangent line to Z, the tact-invariant of the Hessian transformation of λ_x and the dirigent conic will be the tangential equation required; but this tact-invariant will be the discriminant, with respect to k, of the determinant

$$\begin{vmatrix} ka + 2(m^2\lambda_1^2 + 2m\lambda_2\lambda_3), & kh - (\lambda_3^2 + 2m^2\lambda_1\lambda_2), & kg - (\lambda_2^2 + 2m^2\lambda_3\lambda_1) \\ kh - (\lambda_3^2 + 2m^2\lambda_1\lambda_2), & kb + 2(m^2\lambda_2^2 + 2m\lambda_3\lambda_1), & kf - (\lambda_1^2 + 2m^2\lambda_2\lambda_3) \\ kg - (\lambda_2^2 + 2m^2\lambda_3\lambda_1), & kf - (\lambda_1^2 + 2m^2\lambda_2\lambda_3), & kc + 2(m^2\lambda_3^2 + 2m\lambda_1\lambda^2) \end{vmatrix}$$

$$= 0; \tag{100}$$

or, expanded, of the invariant equation

$$A_{111} + kA_{112} + k^2A_{122} + k^3A_{222} = 0, \tag{101}$$

where A_{111} is the discriminant of the Hessian transformation of λ_x, and is $= -2K^2$ (*see* Art. 54). $A_{112} = a\epsilon_{11} + b\epsilon_{22} + c\epsilon_{33} + 2f\epsilon_{23} + 2g\epsilon_{31} + 2h\epsilon_{12}$ (*see* Art. 59); or, the condition the dirigent conic should be conjugate to the transformation of λ_x; and A_{122} is the condition that the latter conic should be conjugate to the former. Hence the required discriminant is

$$(A_{122}^3 + 9K^2A_{112}\cdot A_{122} - 27K^2A_{222}^2)^2 = (A_{122}^2 - 3A_{112}A_{222})^3; \tag{102}$$

and this is the tangential equation of the quartic.

126. The equation (102) is of the twelfth class, and from its form we see that it has twenty-four points of inflection, the tangents at which touch the curve of fourth class

$$A_{122}^2 - 3A_{112}\cdot A_{222} = 0. \tag{103}$$

127. It is plain that if the line λ_x be a double tangent to the quartic, its Hessian transformation will have double contact with the dirigent conic. Now, denoting the Hessian transformation by E, and the dirigent conic by S; then, if E has double contact with S, it will be possible to get a value of k that will make $E + kS$ a perfect square, and any line,

such as λ_x, will meet $E + kS$ in two coincident points. Hence, forming the tangential equation of $E + kS$, let it be

$$F_{11} + 2kF_{12} + k^2F_{22} = 0.$$

F_{11} is the condition that λ_x should touch E. Hence, it is the tangential equation of the cubic U. F_{12} is the condition that λ_x should meet E and S in a harmonic system of points, and is of the fourth degree in the line co-ordinates $\lambda_1, \lambda_2, \lambda_3$. Lastly, F_{22} is the tangential equation of the dirigent conics. Now the value of k, for which

$$F_{11} + 2kF_{12} + k^2F_{22} = 0,$$

is a double root of the equation (101). Hence, eliminating k between the former equation and the two differentials of the latter, and replacing A_{111} by $-2k^2$, we get the determinant

$$\begin{vmatrix} F_{11}, & 2F_{12}, & F_{22} \\ A_{112}, & 2A_{122}, & 2A_{222} \\ -6k^2, & 2A_{112}, & A_{122} \end{vmatrix} = 0. \tag{104}$$

Now, since A_{112}, A_{122}, A_{222} are of the degrees 4, 2, 0, respectively, in line co-ordinates, *this equation is of the tenth class, and represents a curve touching the twenty-eight bi-tangents of the quartic.*

128. If λ_x be an inflectional tangent to the quartic, the Hessian transformation of λ_x must osculate the dirigent conic. Now, the conditions that the Hessian transformation of λ_x should osculate the dirigent conic are, that the equation (101) should be a perfect cube, and this is easily seen to be equivalent to the co-existence of the system of determinants

$$\begin{vmatrix} 3A_{111}, & A_{112}, & A_{122} \\ A_{112}, & A_{122}, & 3A_{222} \end{vmatrix} = 0. \tag{105}$$

Now, this system has twenty-four common tangents, for the curves represented by the determinants

$$\begin{vmatrix} 3A_{111}, & A_{112} \\ A_{112}, & A_{122} \end{vmatrix} = 0; \quad \begin{vmatrix} 3A_{111}, & A_{122} \\ A_{112}, & 3A_{222} \end{vmatrix} = 0, \tag{106}$$

being of the eighth and sixth class, respectively, have forty-eight common tangents, but from this number must be subtracted the twenty-four common tangents of the curves A_{111}, A_{112}, which satisfy the determinants (106), but do not satisfy the determinant

$$\begin{vmatrix} A_{112}, & A_{122}, \\ A_{122}, & 3A_{222}. \end{vmatrix}$$

Hence, there are twenty-four positions of the line λ_x, *such that its Hessian transformation will osculate a given conic; and, therefore, twenty-four positions of* λ_x, *in which it will be an inflectional tangent to the quartic.*

129. We have in Art. 127 given the equation of a curve of the tenth class, which touches all the bi-tangents of a quartic. We shall in this Article determine their numbers. Since the dirigent conic meets the Hessian of the fundamental cubic in six points, and the polar conic of each point breaks up into a pair of lines, we thus get twelve bi-tangents. We account for the remaining sixteen as follows. The equation of the generating conic is

$$y_1U_1 + y_2U_2 + y_3U_3 = 0;$$

where y is a point on the dirigent conic, and U_1, U_2, U_3 are differential co-efficients. Now, it is evident that this conic can in various ways be written in the form

$$t^2X + 2tY + LM = 0; \tag{107}$$

where X and Y are functions of the second degree, and L, M denote lines. Now, the discriminant with respect to t, of the equation (107), gives the quartic; but this is evidently the same as the discriminant with respect to t of the curve

$$t^2LX + 2tY + M = 0; \tag{108}$$

but this equation is a cubic, and if it break up into a line and a conic, the line must be a bi-tangent to the envelope. Now, the condition that a cubic should break up into a line and a conic is of the eighth degree in the coefficients. Hence, if the equation (108) break up into a line and a conic, the condition will involve t in the sixteenth degree. Hence, we get in this manner sixteen bi-tangents, therefore the total number is twenty-eight.

130. If the dirigent conic be the Hessian transformation of λ_x, then, evidently, the quartic must consist of this line itself and a cubic, viz., the companion polar of λ_x (*see* Salmon's *Higher Curves*, page 355). The condition just stated furnishes the conditions that must be fulfilled in order that a quartic should break into a line and a cubic: namely, the dirigent conic must be the Hessian transformation of some line with respect to the cubic. Hence, if the conic be

$$(a, b, c, f, g, h)(x_1, x_2, x_3)^2 = 0,$$

we have the following six equations, the co-existence of which will give the required conditions:—

$$\begin{aligned} 2(m^2\lambda_1^2 + 2m\lambda_2\lambda_3) &= a, & -(\lambda_1^2 + 2m^2\lambda_2\lambda_3) &= f, \\ 2(m^2\lambda_2^2 + 2m\lambda_3\lambda_1) &= b, & -(\lambda_2^2 + 2m^2\lambda_3\lambda_1) &= g, \\ 2(m^2\lambda_3^2 + 2m\lambda_1\lambda_2) &= c, & -(\lambda_3^2 + 2m^2\lambda_1\lambda_2) &= h. \end{aligned}$$

From these we get the following system, in which S denotes the quartic invariant $m - m^4$:—

$$\begin{aligned} 2S\lambda_1^2 &= -m(am + 2f), & 4S\lambda_2\lambda_3 &= a + 2m^2f, \\ 2S\lambda_2^2 &= -m(bm + 2g), & 4S\lambda_3\lambda_1 &= b + 2m^2g, \\ 2S\lambda_3^2 &= -m(cm + 2h), & 4S\lambda_1\lambda_2 &= c + 2m^2h. \end{aligned}$$

Hence,

$$\begin{aligned} 4m^2(am + 2f)(bm + 2g) &= (a + 2m^2f)^2, \\ 4m^2(bm + 2g)(cm + 2h) &= (b + 2m^2g)^2, \\ 4m^2(cm + 2h)(am + 2f) &= (c + 2m^2h)^2. \end{aligned}$$

Hence,

$$\begin{aligned} &8m^3(am + 2f)(bm + 2g(cm + 2h) \\ &= \pm(a + 2m^2f)(b + 2m^2g)(c + 2m^2h), \end{aligned} \tag{109}$$

an equation of the twelfth degree in m. Hence, twelve cubics of the syzygetic pencil

$$x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0,$$

will give quartics that will break up into a line and a cubic.

SECTION II.—COMPANION POLES.

131. If we take the Hessian transformation of λ_x for the dirigent conic, the equation of the quartic Z will be

$$(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{23}, \epsilon_{31}, \epsilon_{12})(U_1, U_2, U_3)^2 = 0; \qquad (110)$$

where ϵ_{11}, ϵ_{22}, &c., have the values given in equations (41), Art. 59. Hence, substituting these values, we get

$$Z \equiv 2mK(4m\lambda_1, 4m\lambda_2, 4m\lambda_3, \lambda_1, \lambda_2, \lambda_3)(U_1, U_2, U_3)^2$$
$$+ (1+8m^3)(\kappa_{11}, \kappa_{22}, \kappa_{33}, \kappa_{23}, \kappa_{31}, \kappa_{12})(\lambda_1 U_1, \lambda_2 U_2, \lambda_3 U_3)^2.$$

Now, restoring the values of U_1, U_2, U_3 in terms of x_1, x_2, x_3, and putting κ_x^2 for the Cayleyan transformation of λ_x (see Art. 62, equation (42), we get, after some calculation,

$$m(4m\lambda_1\ 4m\lambda_2, 4m\lambda_3, \lambda_1, \lambda_2, \lambda_3)(U_1, U_2, U_3)^2 + 4H\lambda_x$$
$$= 4(1+8m^3)\{2x_1x_2x_3\,\lambda_x + m(\lambda_1 x_2^2x_3^2 + \lambda_2 x_3^2x_1^2 + \lambda_3 x_1^2x_2^2)\},$$

and

$$(\kappa_{11}, \kappa_{22}, \kappa_{33}, \kappa_{23}\ \kappa_{31}, \kappa_{12})(\lambda_1 U_1, \lambda_2 U_2, \lambda_3 U_3)^2 - \kappa_x^2(\lambda_x)^2$$
$$= -4K\{2x_1x_2x_3\,\lambda_x + m(\lambda_1 x_2^2x_3^2 + \lambda_2 x_3^2x_1^2 + \lambda_3 x_1^2x_2^2)\}.$$

Hence,

$$Z = (1+8m^3)(\lambda_x)^2\cdot\kappa_x^2 - 8KH\lambda_x. \qquad (111)$$

132. If we take two consecutive points in the dirigent conic, and take the polar conic of each point, the points of intersection of these polar conics will be the four poles of the tangent line joining the two consecutive points. Hence, the quartic Z is the locus of the four poles of all the tangents to the dirigent conic; but, in the case considered in the last Article, one of these poles is a point on the line λ_x, and we see from equation (111) that the locus of the three remaining poles is the cubic

$$(1+8m^3)\lambda_x\kappa_x^2 - 8KH = 0. \qquad (112)$$

We shall call this cubic the locus of the companion poles of all the points on the line λ_x, and we shall denote it by Ω'.

133. If the equation (110) be written in full, it is the determinant

$$\begin{vmatrix} 2(m^2\lambda_1^2 + 2m\lambda_2\lambda_3) & -(\lambda_3^2 + 2m^2\lambda_1\lambda_2), & -(\lambda_2^2 + 2m^2\lambda_3\lambda_1), & U_1 \\ -(\lambda_3^2 + 2m^2\lambda_1\lambda_2), & 2(m^2\lambda_2^2 + 2m\lambda_3\lambda_1), & -(\lambda_1^2 + 2m^2\lambda_2\lambda_3), & U_2 \\ -(\lambda_2^2 + 2m^2\lambda_3\lambda_1), & -(\lambda_1^2 + 2m^2\lambda_2\lambda_3), & 2(m^2\lambda_3^2 + 2m\lambda_1\lambda_2), & U_3 \\ U_1, & U_2, & U_3 & \end{vmatrix} = 0. \quad (113)$$

Now, if the line λ_x be a tangent to the Cayleyan, the matrix of this determinant vanishes. Hence, the determinant is a perfect square, and from equation (111) we see that it reduces, in this case, to

$$(1 + 8m^3)(\lambda_x^2)\,\kappa_x^2 = 0.$$

Hence κ_x^2, or the Cayleyan transformation of λ_x, is a perfect square. This also follows from the fact that the Cayleyan transformation of λ_x is the bordered Cayleyan; and, since its matrix vanishes, it must be a perfect square: let it be the square of μ_x. Hence, omitting constant multipliers, the determinant (113) reduces, in this case, to

$$(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)^2 (\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3)^2 = 0.$$

Again, since λ_x is a tangent to the Cayleyan, its Hessian transformation has a double point: let this double point be called D; then D is a point on the Hessian, and the polar conic of D will, evidently, be the product

$$(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)(\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3) = 0.$$

Hence we have the following important theorem: *If a line λ_x be a tangent to the Cayleyan, the Cayleyan transformation of λ_x is the square of the corresponding tangent.*

134. *The Hessian transformation of the product of the Hessian and any line is equal to the same product multiplied by $-m(1+m^3)$.* For, take the product Hx_1, and writing H in full, we have

$$Hx_1 \equiv -m^2(x_1^2 \cdot x_1^2 + x_1 x_2 \;\, x_2^2 + x_1 x_3 \cdot x_3^2) + (1 + 2m^3)\,x_1^2 \cdot x_2 x_3.$$

Hence, the Hessian transformation is

$$\begin{aligned}-m^2\{(x_2x_3-m^2x_1^2)^2 &+ (m^2x_1x_2-mx_3^2)(x_3x_1-m^2x_2^2)\\ &+ (m^2x_3x_1-mx_2^2)(x_1x_2-m^2x_3^2)\}\\ &+ (1+2m^3)(x_2x_3-m^2x_1^2)(m^2x_2x_3-mx_1^2);\end{aligned}$$

and, reducing, this becomes $-m(1+m^3)Hx_1$. Hence the general proposition is evident.

135. If we perform on λ_x the Cayleyan transformation, and then find the Hessian transformation of the result, we get the determinant

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2, & A_1, \\ \lambda_3, & -2m\lambda_2, & \lambda_1, & A_2, \\ \lambda_2, & \lambda_1, & -2m\lambda_3, & A_3, \\ A_1, & A_2, & A_3, & \end{vmatrix}$$

where A_1, A_2, A_3 have the umbral meanings already explained. Hence, expanding, we get

$$\kappa_{11}\cdot A_{11}+\kappa_{22}\cdot A_{22}+\kappa_{33}\cdot A_{33}+2\kappa_{23}\cdot A_{23}+2\kappa_{31}\cdot A_{31}+2\kappa_{12}\cdot A_{12} = 0;$$

or,

$$\kappa_A^2 = 0. \tag{114}$$

This result might have been foreseen, since the Hessian transformation consists in substituting the symbols A_1, A_2, A_3 for x_1, x_2, x_3; and the Cayleyan in substituting κ_1, κ_2, κ_3 for λ_1, λ_2, λ_3. If we restore the values of the minors κ_{11}, κ_{22}, &c.; A_{11}, A_{22}, &c.; the equation (114) becomes

$$\begin{aligned}&\{3m^2\lambda_1^2+2m(1+2m^3)\lambda_2\lambda_3\}x_1^2+\{(1-4m^3)\lambda_1^2-6m^2\lambda_2\lambda_3\}x_2x_3\\ +&\{3m^2\lambda_2^2+2m(1+2m^3)\lambda_3\lambda_1\}x_2^2+\{(1-4m^3)\lambda_2^2-6m^2\lambda_3\lambda_1\}x_3x_1\\ +&\{3m^2\lambda_3^2+2m(1+2m^3)\lambda_1\lambda_2\}x_3^2+\{(1-4m^3)\lambda_3^2-6m^2\lambda_1\lambda_2\}x_1x_2=0. \quad (115)\end{aligned}$$

Cor.—The Cayleyan, the Hessian, and the Hesso-Cayleyan transformations of λ_x *may be written in the forms* κ_x^2, λ_A^2, κ_A^2, *respectively.*

136. Since, if λ_x be a tangent to the Cayleyan, its Cayleyan transformation κ_x^2 will be the square of the corresponding tangent (see Art. 134); therefore the Hessian transformation of κ_x^2 will denote a pair of lines touching the Hessian at corresponding points (see Art. 37); κ_A^2 will denote a pair of tangents to the Hessian. Hence, *the Hesso-Cayleyan transformation of any tangent to the Cayleyan is a pair of lines touching the Hessian at corresponding points.*

137. Since $Z \equiv (1+8m^3)\lambda_x^2 \cdot \kappa_x^2 - 8KH\lambda_x$ (see Art. 132); then, if we denote the Hessian transformation of Z by Z_h, we get from recent Articles

$$Z_h \equiv (1+8m^3)\lambda_A^2 \cdot \kappa_A^2 + 8m(1+m^3)KH\lambda_x = 0. \tag{116}$$

Now, the Hessian transformation of Z is (*see* Art. 121) the locus of a point whose polar line, with respect to the Hessian, touches the dirigent conic. Hence it follows, from equation (116), that the conics λ_A^2 and κ_A^2 meet the Hessian in the points of contact of common tangents to the Hessian and λ_A^2; but λ_A^2 has triple contact with the Hessian, and the tangent at each contact will count for two common tangents. Hence, the remaining common tangents will touch the Hessian at the six points of intersection of k_A^2 with the Hessian. Hence we have the following theorem: *The six tangents to the Hessian, at its six points of intersection with the Hesso-Cayleyan transformation of* λ_x, *are also tangents to the Hessian transformation of* λ_x.

138. The polar line of every point on λ_x, with respect to the fundamental cubic, is a tangent to the conic λ_A^2, and the polar line of every point on the Cayleyan is a tangent to the Hessian. Therefore the polar lines of the six points of intersection of λ_x with the Cayleyan are common tangents of the conic λ_A^2 and the Hessian, and we have the following theorem:

The six tangents to the Hessian, at its six points of intersection with the Hesso-Cayleyan transformation of λ_x, *are the polar lines with respect to the fundamental cubic of the six points of intersection of* λ_x *with the Cayleyan.*

139. If P be a point on the Hessian, π its corresponding point, Q the point on the Cayleyan, which is the pole with respect to the fundamental cubic of the tangent to the Hessian at P, then π and Q are harmonic conjugates to the two remaining points in which the line πQ intersects the Hessian. Hence we have the following remarkable theorem: *If six tangents be drawn to the Cayleyan at the six points in which the line λ_x intersects it, then each tangent intersects the Hessian in two corresponding points, and in a third point, which is the harmonic conjugate to the point of contact, with respect to the two corresponding points. And the six points, which are the harmonic conjugates, are the points of intersection of the Cayleyan transformation of λ_x with the Hessian.*

140. We may prove, in a manner exactly similar to that of Art. 134, that the Cayleyan transformation of the product of the Cayleyan and any line is the same product multiplied by $-(1-8m^3)$, and that the Cayleyan transformation κ_x^2 is $(1+8m^2)\lambda_x^2$. Hence the Cayleyan transformation of Z, which we may denote by Z_k, is

$$(1+8m^3)^2\lambda_x^2\kappa_x^2 + 8(1-8m^3)KH\lambda_x. \quad (117)$$

Again, Z is the determinant (113); and, forming the Cayleyan transformation of its constituents, we get Z_k equal to the determinant

$$\begin{vmatrix} 6m^2\lambda_1^2+4m(1+2m^3)\lambda_2\lambda_3, & (1-4m^3)\lambda_3^2-6m^2\lambda_1\lambda_2, & (1-4m^3)\lambda_2^2-6m^2\lambda_3\lambda_1, & U_1 \\ (1-4m^3)\lambda_3^2-6m^2\lambda_1\lambda_2, & 6m^2\lambda_2^2+4m(1+2m^3)\lambda_3\lambda_1, & (1-4m^3)\lambda_1^2-6m^2\lambda_2\lambda_3, & U_2 \\ (1-4m^3)\lambda_2^2-6m_2\lambda_3\lambda_1, & (1-4m^3)\lambda_2^2-6m^2\lambda_2\lambda_3, & 6m^2\lambda_3^2+4m(1+2m^3)\lambda_1\lambda_2, & U_3 \\ U_1, & U_2, & U_3 & \end{vmatrix} = 0. \quad (118)$$

Now, the determinant (118) is the quartic we should get as the locus of points whose polar lines, with respect to the fundamental cubic, touch the conic κ_A^2. Hence, we have the following theorem: *The locus of the poles, with respect to the fundamental cubic of all the lines which touch the conic κ_A^2, or the Hesso-Cayleyan transformation of λ_x, is the quartic*

$$(1+8m^3)^2\lambda_x^2\cdot\kappa_x^2 + 8(1-8m^3)KH\lambda_x = 0.$$

141. If the curve Ω' (see Art. 132) pass through a fixed point x', then, substituting the co-ordinates of this point, viz., x'_1, x'_2, x'_3 in the equation of Ω', we get

$$(1+8m^3)\,\kappa_{x'}^{2}\cdot\lambda_{x'} - 8KH' = 0; \qquad (119)$$

where the accents denote the result of substituting the co-ordinates of x' in the expressions κ_x^2, λ_x, and H, respectively. Now, this equation (119) is of the third class, and is evidently the equation of the three points, which, together with x', form the four poles of some line, that is, (119) is the equation of the three companion poles of x'.

142. If the Hessian transformation of λ_x touch the line μ_x, we have the equation of the fourth class,

$$(\epsilon_{11},\ \epsilon_{22},\ \epsilon_{33},\ \epsilon_{23},\ \epsilon_{31},\ \epsilon_{12})(\mu_1,\ \mu_2,\ \mu_3)^2 = 0. \qquad (120)$$

Now, since the line μ_x is a tangent to the Hessian transformation of λ_x, it must be the polar line of some point on λ_x, with respect to the fundamental cubic, and, being of the fourth class, it must be the equation of the four poles of μ_x. Hence, the equation of the four poles is the equation (120), or, expanded,

$$\begin{aligned}&2mK(4m\lambda_1,\ 4m\lambda_2\ 4m\lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3)(\mu_1,\ \mu_2,\ \mu_3)^2\\ &+(1+8m^3)(\kappa_{11},\ \kappa_{22},\ \kappa_{33},\ \kappa_{23},\ \kappa_{31},\ \kappa_{12})(\lambda_1\mu_1,\ \lambda_2\mu_2,\ \lambda_3\mu_3)^2 = 0. \qquad (121)\end{aligned}$$

143. If the line λ_x touch the fundamental cubic, two of the three points of intersection coincide, and the Hessian of the cubic, whose roots are represented by the distances from any arbitrary point to their intersections, must be a perfect square. Hence (see Art. 47), the line λ_x must be a tangent to the Hessian transformation of λ_x; and, therefore, if we replace μ_1, μ_2, μ_3 in equation (110) by λ_1, λ_2, λ_3, we have the tangential equation of the fundamental cubic.

CHAPTER V.

Section I.—Reciprocation.

144. In Art. 63 we have been led to a contravariant Ψ, which has very important relations to the cubic: for the purpose of studying these relations we shall consider the equations of these two curves in point and line co-ordinates, respectively:

$$U \equiv x_1^3 + x_2^3 + x_3^3 + 6m\, x_1 x_2 x_3 = 0. \qquad (122)$$

$$\Psi \equiv m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 = 0. \qquad (123)$$

Now, the Hessian and the Cayleyan of U are

$$-m^2(x_1^3 + x_2^3 + x_3^3) + (1 + 2m^3)\, x_1x_2x_3 = 0,$$

$$m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3 = 0,$$

respectively; and if we form the corresponding contravariant and covariant for Ψ, which is evidently done by interchanging point and line co-ordinates, and changing m into $-\frac{1}{2}m$, we get the very same concomitants, only interchanged. Hence, extending the terms Cayleyan and Hessian, we see that the curves U and Ψ are so related, that the Cayleyan of U is the Hessian of Ψ, and the Hessian of U is the Cayleyan of Ψ. Hence, the properties of Cayleyan and Hessian are reciprocal: and, therefore, from known properties of either of these curves we can, by reciprocation, get properties of the other.

145. From the equation

$$m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (1 - 4m^3)\lambda_1\lambda_2\lambda_3 = K,$$

$$m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) - 3\lambda_1\lambda_2\lambda_3 = \Psi,$$

we get

$$4(1 - m^3)\lambda_1\lambda_2\lambda_3 = K - \Psi;$$

$$4S(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = (1 - 4m^3)\Psi + 3K.$$

Hence, any homogeneous equation in $\lambda_1\lambda_2\lambda_3$ and $\lambda_1^3 + \lambda_2^3 + \lambda_3^3$ can be expressed in terms of Ψ and K. Thus

$$A(\lambda_1\lambda_2\lambda_3^2) + B\lambda_1\lambda_2\lambda_3(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + C(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^2$$

gives, in terms of K and Ψ,

$$(m^2A + 3mB + 9C)K^2 - \{2mA + 2m(1+2m^3)B - 6(1-4m^3)C\}K\Psi$$
$$+ \{m^2A - m(1-4m^3)B + (1-4m^3)^2 C\}\Psi^2. \qquad (124)$$

146. We now give some examples of cubic reciprocation. It will be seen that our methods are virtually identical with the methods of conic sections.

The Hessian of a cubic is the locus of points whose polar conics, or first emanants, with respect to U, break up into factors denoting lines.

Reciprocally: *The Cayleyan is the envelope of lines whose first emanants, with respect to Ψ, break up into factors denoting points.*

147. The Hessian transformation of a line is the locus of points whose polar conics, with respect to U, touch the line.

Reciprocally: *The Cayleyan transformation of a point is the envelope of lines whose first emanants, with respect to Ψ, pass through the given point.*

148. The Hessian transformation of any tangent to the Cayleyan breaks up into a pair of lines touching the Hessian, and also having their point of intersection on the Hessian.

Reciprocally: *The Cayleyan transformation of any point on the Hessian breaks up into two factors denoting points: these points are on the Cayleyan, and their line of connexion is a tangent to the Cayleyan.*

149. The Hessian transformation of any line touches the Hessian in three points, which correspond to the points of intersection of the line with the Hessian.

Reciprocally: *The Cayleyan transformation of any point touches the*

Cayleyan in three points, the three tangents at which form three corresponding tangents with the three tangents drawn from the point to the Cayleyan: that is, one triad of tangents meets the other triad in three points lying on the Hessian.

150. The Hessian transformation of a line, and the Hessian itself, have, besides the tangents at their three points of contact, six other common tangents which touch the Hessian at its six points of intersection with the Hesso-Cayleyan transformation of the line.

Reciprocally: *The Cayleyan transformation of a point is a conic touching the Cayleyan in three points and intersecting it in six other points, the tangents at which to the Cayleyan are also tangents to the Cayleyan transformation of the point.*

151. The Hessian transformation of any tangent to the Hessian is a conic having double contact with the Hessian: namely, a two-pointic and a four-pointic contact, and the envelope of the chord of contact is the Cayleyan.

Reciprocally: *The Cayleyan transformation of any point on the Cayleyan is a conic having double contact with the Cayleyan: that is, a two-pointic and a four-pointic contact; and the locus of the pole of the chord of contact, with respect to the conic, is the Hessian.*

152. The Hessian transformation of any inflectional tangent to the Hessian is a conic, having six-pointic contact at the corresponding point.

Reciprocally: *The Cayleyan transformation of any of the nine cusps is a conic, having six-pointic contact with the Cayleyan; and the tangent at the point of contact is the tangent corresponding to the cuspidal tangent.*

153. The Cayleyan transformation of λ_x meets the Hessian in six points where the tangents to the Cayleyan, at its points of intersection with the line λ_x, also meet it; and the point of intersection and point

of contact on each tangent are harmonic conjugates to the two remaining points in which the same tangent meets the Hessian (see Art. 140).

Reciprocally: *The six common tangents of the Cayleyan and the conic, which is the Hessian transformation of the line λ_x, meet the Hessian at the points of contact of the six tangents from the point λ_x to the Hessian; and the tangent at any of these points to the Hessian forms a harmonic pencil with the three tangents from the same point to the Cayleyan.*

154. If the polar conic of a point A has its centre at B, then the polar of B has its centre at A.

Reciprocally: *If the polar conic of a line α, with respect to the curve Ψ, has the line β for diameter, then the polar conic of β has α for diameter.*

155. If a triangle be formed by joining the three points which correspond to the three points at infinity on the Hessian, the polar conics of all the points on any side of this triangle are concentric, their common centre being the opposite angular point.

Reciprocally: *If from any assumed point A three tangents be drawn to the Cayleyan, their three corresponding tangents form a triangle whose three angular points B, C, D form with A a quadrangle or tetrastigm, such that the triangle formed by any three will be self-conjugate, with respect to the polar conics, of all lines passing through the fourth.*

Cor. 1.—*The system of six tangents are the six lines joining the four points A, B, C, D.*

Cor. 2.—*The points A, B, C, D lie on the Hessian.*

Cor. 3.—*The equation of Ψ expressed in terms of the equation of A, B, C, D will be of the form*

$$\alpha_1\lambda_1^3 + \alpha_2\lambda_2^3 + \alpha_3\lambda_3^3 + \alpha_4\lambda_4^3 = 0; \tag{125}$$

and the corresponding Cayleyan will be

$$(\alpha_1\lambda_1)^{-1} + (\alpha_2\lambda_2)^{-1} + (\alpha_3\lambda_3)^{-1} + (\alpha_4\lambda_4)^{-1} = 0. \tag{126}$$

156. If the polar conic of a point A, with respect to U, has a double point at B, the polar conic of B has a double point at A. This theorem is evidently a special case of the theorem of Art. 154. Also, the line AB is a tangent to the Cayleyan, A and B are points on the Hessian, and their polar conics have each double contact with the Cayleyan.

Reciprocally: *If the first emanant of a line λ_x, with respect to Ψ, break up into two points P, P' lying on the line μ_x, then, the first emanant of μ_x, with respect to Ψ, breaks up into two points Q, Q' lying on λ_x; the lines λ_x, μ_x are tangents to the Cayleyan; their point of intersection lies on the Hessian. The four points P, P', Q, Q' also lie on the Hessian, and form on it two pairs of corresponding points.*

157. Any three tangents, AN, AR, AB (see Fig., Art. 159) to the Cayleyan, from any point A of the Hessian, form, with the tangent AD at A to the Hessian, a harmonic pencil.

Reciprocally: *Any tangent to the Cayleyan meets the Hessian in three points, which, with the point of contact on the Cayleyan, form a harmonic system of points.*

158. If from the points where any line cuts the Hessian three pairs of corresponding tangents be drawn to the Cayleyan, any seventh tangent to the Cayleyan will be cut in involution by the three pairs of corresponding tangents, and the locus of the double points of the involution will be the Hessian.

Reciprocally: *If from any point on the Hessian a pencil of six lines be drawn to three pairs of corresponding points, the pencil will be in the involution, and the envelope of its double lines will be the Cayleyan.*

We may give a direct proof of the theorem of this Article very simply as follows:—

Let a, a'; b, b'; c, c' be three pairs of points: then, denoting the angle formed at any point P by any two points a, b by $(\widehat{ab})$, the condition of the involution gives us

$$\sin(\widehat{ab'})\sin(\widehat{bc'})\sin(\widehat{ca'}) = k\sin(\widehat{a'b})\sin(\widehat{b'c})\sin(\widehat{c'a}).$$

Hence, denoting the normal form of the equation of the line joining two points a, b by (ab), we get

$$(ab')(bc')(ca') = k(a'b)(b'c)(c'a), \tag{127}$$

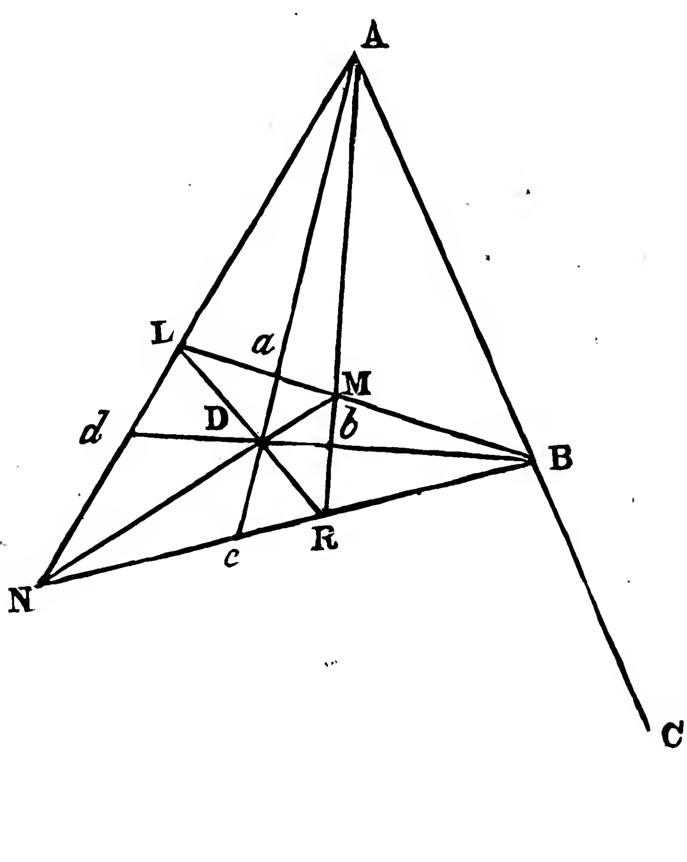
when k is a constant. Hence, the locus of P is a curve of the third order.

159. If A and B be corresponding points on the Hessian, the product of the equations of the tangents at A and B is the Hessian transformation of the equation of the line joining the points A, B.

Reciprocally: *If α and β be two corresponding tangents to the Cayleyan, the product of the equations of their points of contact is the Cayleyan transformation of the equation of the point of intersection of the lines α and β.*

Or, we may state it thus:—

Let the line λ_x be a tangent to the Cayleyan, and let it be the line ABC, meeting the Hessian in the points A, B, C. A and B being the corresponding points, and C the third point, let the polar conic of A be the lines BL, BN, and the polar conic of B the lines AR, AN. Then D, the intersection of the diagonals of the quadrilateral $LMRN$, is the point corresponding to C; and the polar conic of C is the pair of lines LR and MN. Then (see Art. 36), we have the following



theorem: If λ_x be the equation of the line AB, $\lambda_A{}^2$ is the equation of the pair of lines AD, BD. Then, reciprocally, if $\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$ be the equation of the point of intersection of two corresponding tangents to the Cayleyan, the equation of their points of contact is the determinant

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2, & \xi_1 \\ \lambda_3, & -2m\lambda_2, & \lambda_1, & \xi_2 \\ \lambda_2, & \lambda_1, & -2m\lambda_3, & \xi_3 \\ \xi_1, & \xi_2, & \xi_3 & \end{vmatrix} = 0. \qquad (128)$$

160. Any tangent to the Cayleyan intersects it in four points. One of the three involutions determined by these four points has for double points the pair of corresponding points in which the same line meets the Hessian; and, for a pair of homologous points, the point of contact on the Cayleyan and the third point where it meets the Hessian.

Reciprocally: *If from any point P on the Hessian four tangents be drawn to the Hessian, the corresponding pair of tangents drawn from the same point to the Cayleyan form the double lines of one of the three involutions determined by the four Hessian tangents; also, the tangent at the point P and the third tangent drawn from P to the Cayleyan form a pair of homologous lines of the same involution.*

161. If $\lambda_x = 0$ be the equation of any line A, the equation of the three lines B, C, D, which together with A make a system of four conjugate lines, with respect to U, is (see Art. 95)

$$KH - S\lambda_A{}^2\lambda_x = 0. \qquad (129)$$

Hence, reciprocally: If λ_x be the equation of any point A, the equation of the three points B, C, D, which together with A make a conjugate tetrastigm, with respect to Ψ, will be obtained (see Art. 144) by interchanging point and line co-ordinates, and putting $-\frac{1}{2}m$ in place of m: in this way we get from equation (129) the equation

$$(1 + 8m^3)\lambda_x\kappa_x{}^2 - 8KH = 0. \qquad (130)$$

Cor.—If we compare the equation (130) with equation (112), they are the same in form but have different significations, which may be explained thus: in equation (112) λ_1, λ_2, λ_3 are constants, and x_1, x_2, x_3 are variables: in (130) the x's are constant and the λ's variable. In fact the equation is a Zwischenform, and represents, as all Zwischenformen do, different curves according as it is considered in point or line co-ordinates.

162. The polar conic of D (see Fig., Art. 159), with respect to U, is two lines intersecting in C. Let these lines be λ_x, λ'_x, then the Hessian transformation of their product will be the polar conic of D, with respect to the Hessian (see Art. 42, *Cor.*), viz., this is

$$\begin{vmatrix} x_1, & mx_3, & mx_2, & \lambda_1 \\ mx_3, & x_2, & mx_1, & \lambda_2 \\ mx_2, & mx_1, & x_3, & \lambda_3 \\ \lambda'_1, & \lambda'_2, & \lambda'_3 & \end{vmatrix} = 0, \qquad (131)$$

and the conic passes through the six points on the Hessian corresponding to the points where λ_x, λ'_x intersect the Hessian; or, in other words, λ_x, λ'_x form one of the three pairs of lines joining the points of contact on the Hessian of the tangent from D.

Reciprocally: *If λ_ξ, $\lambda_{\xi'}$ be the equation of two corresponding points on the Hessian, the Cayleyan transformation of their product will be the polar conic, with respect to the Cayleyan, of the tangent to the Cayleyan which corresponds to the line joining these points*, viz., this transformation is got from equation (131), by interchanging point and line co-ordinates, and changing m into $-\frac{1}{2}m$, thus:

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2, & \xi_1 \\ \lambda_3, & -2m\lambda_2, & \lambda_1, & \xi_2 \\ \lambda_2, & \lambda_1, & -2m\lambda_3, & \xi_3 \\ \xi'_1, & \xi'_2, & \xi'_3 & \end{vmatrix}. \qquad (132)$$

The equation thus written has another signification, which we explain thus:

Let us consider the reciprocal of the theorem, that if two lines be tangents to the Hessian at corresponding points, the Hessian transformation of their product will be a perfect square, and we get the following theorem: *If* $\lambda_\xi = 0$, $\lambda_\xi' = 0$ *be the equations of the points of contact of two corresponding tangents to the Cayleyan, then the Cayleyan transformation of the product of these equations is a perfect square, and will denote the square of the equation of the point of intersection of these tangents.* The difference lies in the positions of the points λ_ξ, λ_ξ' in the two theorems.

163. We have seen, Art. 133, that if the line λ_x be a tangent to the Cayleyan, the square of its corresponding tangent will be the Cayleyan transformation of the function λ_x.

Reciprocally: If $\lambda_\xi = 0$ *be the equation of any point on the Hessian, the square of its corresponding point is the Hessian transformation of* λ_ξ, *viz., this is*

$$\begin{vmatrix} \xi_1, & m\xi_3, & m\xi_2, & \lambda_1, \\ m\xi_3, & \xi_2, & m\xi_1, & \lambda_2, \\ m\xi_2, & m\xi_1, & \xi_3, & \lambda_3, \\ \lambda_1, & \lambda_2, & \lambda_3, & \end{vmatrix}. \qquad (133)$$

The theorem of this Article is very important: it will be the fundamental theorem of Chapter VII.

164. We have in equation (120), Art. 142, given the equation of the four poles of the line μ_x, with respect to the fundamental cubic. Hence, by reciprocation, we get the equation of the four polars of the point μ_ξ, with respect to the curve Ψ, viz.,

$$(\kappa_{11}, \kappa_{22}, \kappa_{33}, \kappa_{23}, \kappa_{31}, \kappa_{12})(\xi_1, \xi_2, \xi_3)^2 = 0; \qquad (134)$$

where κ_{11}, κ_{22}, &c., denote the result of substituting x_1, x_2, x_3 for the line co-ordinates in the minors of the Cayleyan.

Cor.—If the point μ_ξ be on Ψ, we may replace ξ_1, ξ_2, ξ_3 by x_1, x_2, x_3, and we have the equation of Ψ in point co-ordinates.

165. In Art. 141 we see that the equation $(1+8m^3)\kappa_x^2 \cdot \lambda_x - 8KH = 0$ denotes, if x be a fixed point, that is if x_1, x_2, x_3 be fixed co-ordinates, the tangential equation of the three points which are the companion poles of x. Hence, by the usual mode of reciprocation: *If λ_x be the equation of any line, the equation of its three companion lines, with respect to the cubic Ψ, will be* $KH - S\lambda_A^2\lambda_x = 0$.

"Companion lines" are lines which, with λ_x, form in line co-ordinates the four polar lines of a point.

166. The Zwischenform $KH - S\lambda_A^2\lambda_x$ has four interpretations, which may be stated as follows: Two with respect to U, and two with respect to Ψ.

Interpretations with respect to U:—

1°. *If the line co-ordinates be constant, the equation* $KH - S\lambda_A^2\lambda_x = 0$ *denotes the three lines which, with λ_x, form a conjugate tetragram, with respect to the cubic; or, in other words, it denotes the sides of the triangle joining the three points which correspond to the three points where λ_x intersects the Hessian.*

2°. *If the point co-ordinates be constant, the same equation denotes the envelope of three sides of a self-conjugate tetragram, when the fourth side passes through a given point.*

Interpretation with respect to Ψ:—

1°. *If the line co-ordinates be constant, the equation denotes the three companion lines which, with λ_x, are the four polar lines of a fixed point.*

2°. *If the point co-ordinates be constant, the equation is the envelope of three companion polar lines of a fourth which passes through a given point.*

Cor. 1.—In the same manner, it may be shown that the Zwischenform $(1+8m^3)\kappa_x\lambda_x - 8KH$ has four interpretations.

Cor. 2.—*If four lines form a self-conjugate tetragram with respect to U, they are four companion polars with respect to Ψ.*

Cor. 3.—*If four points form a self-conjugate tetrastigm with respect to Ψ, they are four conjugate poles with respect to U.*

167. From Art. 101, *Cor.* 2, we see that the locus of a point, the central transformation of whose polar line, with respect to π, passes through the point, is the Zwischenform

$$\frac{d\pi}{da_1}\cdot\frac{d\pi}{dx_1} + \frac{d\pi}{da_2}\cdot\frac{d\pi}{dx_2} + \frac{d\pi}{da_3}\cdot\frac{d\pi}{dx_3} = 0.$$

Now, denoting this form by F for the present, we see that if A be the point, and B its corresponding point, and L the polar line of A with respect to π, then, because the central transformation of L passes through A, the point B must be on L. Hence, the polar line of B with respect to π passes through A. Hence B must be a point on the locus: in other words, the central transformation of F must contain F as a factor. And in fact it is easily verified, by calculation, that the central transformation of F is ΩF. Hence we have the following theorem:—*The central transformation of F reproduces the curve F.*

This theorem properly belongs to Chapter III.

CHAPTER VI.

Section I.—Zwischenformen Conics.

168. We have seen that the Hessian transformation of the line λ_x, with respect to the fundamental cubic, is

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \lambda_1 \\ u_{21}, & u_{22}, & u_{23}, & \lambda_2 \\ u_{31}, & u_{32}, & u_{33}, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3 & \end{vmatrix} = 0.$$

Hence the Hessian transformation of λ_x, with respect to $U + kV$, is

$$\begin{vmatrix} u_{11} + kv_{11}, & u_{12} + kv_{12}, & u_{13} + kv_{13} & \lambda_1 \\ u_{21} + kv_{21}, & u_{22} + kv_{22}, & u_{23} + kv_{23}, & \lambda_2 \\ u_{31} + kv_{31}, & u_{32} + kv_{32}, & u_{33} + kv_{33}, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3 & \end{vmatrix} = 0. \qquad (135)$$

169. We shall use the following notation in this Chapter: viz., we shall put

$$P \text{ for } \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \lambda_3^2 x_3^2 - 2\lambda_2\lambda_3 x_2 x_3 - 2\lambda_3\lambda_1 x_3 x_1 - 2\lambda_1\lambda_2 x_1 x_2,$$

$$Q \text{ ,, } \lambda_2\lambda_3 x_1^2 + \lambda_3\lambda_1 x_2^2 + \lambda_1\lambda_2 x_3^2,$$

$$R \text{ ,, } \lambda_1^2 x_2 x_3 + \lambda_2^2 x_3 x_1 + \lambda_3^2 x_1 x_2.$$

Now in the determinant (135), if U denote the fundamental cubic, and V its Hessian, the expansion of the determinant gives

$$E_1 + 2kE_2 + k^2 E_3 = 0; \qquad (136)$$

where

$$E_1 \equiv m^2 P + 2m Q - R,$$

$$E_2 \equiv m(1+2m^3)P + (1-4m^3)Q + 6m^2 R,$$

$$E_3 \equiv (1+2m^3)^2 P + 12m^2(1+2m^3)Q - 36m^4 R.$$

170. In a manner exactly the same, if we form the Cayleyan transformation of λ_x, with respect to $\Psi + kK$, where Ψ is the cubic of Art. 63, we get the three following Zwischenforms :—

$$E_4 \equiv -P + 4m^2 Q + 4mR,$$

$$E_5 \equiv (1-4m^3)P + 12m^2 Q + 4m(1+2m^3)R,$$

$$E_6 \equiv (1-4m^3)^2 P - 36m^2 Q + 12m(1-4m^3)R.$$

These three forms may also be derived from the forms E_1, E_2, E_3 by the transformation in Art. 144, Chapter V.

171. By the methods of Chapter II., section 1, we get the Hessian transformation of the conics P, Q, R: they are given in the following Table:—Thus the Hessian

$$\begin{array}{ll} \text{transformation of } P \equiv & \left| \begin{array}{cccc} 0, & 2m, & 1, & -m^2, \\ -1, & -4m^2, & 0, & 1, \\ -m, & 0, & 2m^2, & -m, \end{array} \right| (P,\ Q,\ R,(\lambda_x)^2). \\ \quad\text{,,} \qquad\qquad\quad 4Q \equiv & \\ \quad\text{,,} \qquad\qquad\quad 2R \equiv & \end{array} \qquad (137)$$

$(\lambda_x)^2$ denotes in these expressions the square of the function

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0.$$

From these equations we can find the condition that the Hessian transformation of $aP + bQ + cR$ may be of the form $a'P + b'Q + c'R$; that is, that it may not contain the term $(\lambda_x)^2$ to be $4am^2 - b + 2amc = 0$.

In this manner we get the two following fundamental forms, viz.:

$$E_7 \equiv 3m^2 P + 2m(1+2m^3)\,Q + (1-4m^3)\,R,$$

$$E_8 \equiv -(5m^2+4m^5)\,P + 2m(1-10m^3)\,Q + (1+8m^6)\,R.$$

The first of these is the Hessian transformation of E_4, and the next of E_5.

172. We can form in a similar manner the Cayleyan transformations of P, Q, R; we give them here. Thus the Cayleyan

$$\begin{array}{ll} \text{transformation of } P \equiv & \left| \begin{array}{cccc} 0, & 4m^2, & -4m, & -1, \\ m, & 1, & 0, & m, \\ -m^2, & 0, & -1, & m^2, \end{array} \right| (P,\ Q,\ R(\lambda_x)^2). \\ \quad\text{,,} \qquad\qquad\quad Q \equiv & \\ \quad\text{,,} \qquad\qquad\quad R \equiv & \end{array} \qquad (138)$$

Hence, if $a - mb - m^2c = 0$, the Cayleyan transformation of $aP + bQ + cR$ will be of the same form. In this way we find the Cayleyan transformation of E_1 is the same as the Hessian transformation of E_4, namely, E_7, viz.: this is the Hesso-Cayleyan transformation of λ_x. And we find

$$E_9 \equiv m(1-10m^3)\,P + (1+8m^6)\,Q - (10m^2+8m^5)\,R:$$

this is the Cayleyan transformation of E_2.

173. The following Table contains the equations of our nine Zwischenform conics:—

$$
\begin{array}{ll}
E_1 \equiv & \left| \begin{array}{ccc} m^2, & 2m, & -1, \\
E_2 \equiv & m(1+2m^3), & 1-4m^3, & 6m^2, \\
\end{array}\right.
\end{array}
$$

$$
\begin{array}{rl}
\begin{array}{r} E_1 \equiv \\ E_2 \equiv \\ E_3 \equiv \\ E_4 \equiv \\ E_5 \equiv \\ E_6 \equiv \\ E_7 \equiv \\ E_8 \equiv \\ E_9 \equiv \end{array}
&
\left|\begin{array}{ccc}
m^2, & 2m, & -1, \\
m(1+2m^3), & 1-4m^3, & 6m^2, \\
(1+2m^3)^2, & 12m^2(1+2m^3), & -36m^4, \\
-1, & 4m^2, & 4m, \\
(1-4m^3), & 12m^2, & 4m(1+2m^3), \\
(1-4m^3)^2, & -36m^2, & 12m(1-4m^3), \\
3m^2, & 2m(1+2m^3), & (1-4m^3), \\
-(5m^2+4m^5), & 2m(1-10m^3), & 1+8m^6, \\
m(1-10m^3), & (1+8m^6), & -(10m^2+8m^5).
\end{array}\right| (P,\ Q,\ R).
\end{array}
\qquad (139)
$$

174. If we write the fundamental cubic a_x^3 and its Hessian α_x^3, then it is evident, from the method by which we arrived at E_2, that E_2 is obtained from E_1 as follows: differentiate with respect to the co-efficients a_{ikh} of a_x^3, multiply by the corresponding co-efficients of α_x^3, and add the results. Clebsch denotes this process by the letter δ, thus $\delta E_1 = E_2$. In like manner $\delta E_2 = E_3 + \frac{1}{2} SE_1$, and $\delta E_3 = SE_2$.

175. E_2 is the locus of the points whose polar conics with respect to the cubic and its Hessian are cut harmonically by the line λ_x, and E_3 is related to the Hessian in the same manner as E_1 is to the fundamental cubic. Hence E_3 has triple contact with the Hessian of the Hessian, or, say, the second Hessian. Hence if λ_x passes through a fixed point, the envelope of E_3 will be the second Hessian multiplied by the polar line of the point with respect to the first Hessian.

176. The following Table contains the Hessian transformations of the conics E_1, E_2, E_3, E_6 :—

$$\begin{array}{l} \text{Of } E_1 \equiv \\ \;,,\; 4E_2 \equiv \\ \;,,\; E_3 \equiv \\ \;,,\; E_6 \equiv \end{array} \left| \begin{array}{llll} 0, & 0, & 0, & S, \\ -(1+8m^3), & 4m^2(1+8m^3), & 4m(1+8m^3), & T, \\ 3m^2(1+8m^3), & 2m(1+8m^3)(1+2m^3), & (1+8m^3)(1-4m^3), & -4S^2, \\ 3m^2(1+8m^3), & 2m(1+8m^3)(1+2m^3), & (1+8m^3)(1-4m^3), & -16S^2, \end{array} \right| (P, Q, R(\lambda_x)^2), \qquad (140)$$

where S and T denote as usual the quartic and sextic invariants of the cubic.

178. The Cayleyan transformations are—

$$\begin{aligned} \text{Of } E_3 &\equiv -4m(1-m^3)E_7 - (1+8m^3)^2(\lambda_x)^2, \\ ,,\; E_4 &\equiv (1+8m^3)(\lambda_x)^2, \\ ,,\; E_5 &\equiv 8SE_1 - T(\lambda_x)^2, \\ ,,\; E_6 &\equiv 16SE_7 - (1+8m^3)^2(\lambda_x)^2. \end{aligned} \qquad (141)$$

178. From the Table 129, we get several identities, thus :—

$$\begin{aligned} 4SE_1 + E_3 + (1+8m^3)E_4 &= 0, \\ 16SE_1 + E_6 + (1+8m^3)E_4 &= 0, \\ TE_1 - 4SE_2 - (1+8m^3)E_7 &= 0, \\ TE_4 + (1+8m^3)E_5 - 8SE_7 &= 0, \\ 4E_3 - E_6 + 3(1+8m^3)E_4 &= 0, \\ 12SE_1 + E_6 - E_3 &= 0, \\ TE_7 + (1+8m^3)^2 E_1 - 4SE_9 &= 0, \qquad (142) \\ TE_7 + 8m^2(1-m^3)^2 E_4 - (1+8m^3)E_8 &= 0, \\ 8STE_1 - 32S^2E_2 &= (1+8m^3)TE_4 + (1+8m^3)^2 E_5, \\ 12STE_1 - 32S^2E_2 + TE_3 &= 1+8m^3)^2 E_5. \end{aligned}$$

179. From the foregoing many inferences may be drawn: thus, from the two first we see that the four conics E_1, E_3, E_4, E_6, considered as conics in point co-ordinates, pass through the same four points; and, as conics in line co-ordinates, they are inscribed in the same quadrilateral. The same properties hold for the two systems

$$E_1, \quad E_2, \quad E_7, \quad E_9;$$

$$E_4, \quad E_5, \quad E_7, \quad E_8.$$

Again we see, from equations (141), that the Cayleyan transformation of E_3, E_5, E_6 have double contact with the conics E_7, E_1, E_7 respectively, so that the Cayleyan transformation of E_3 has double contact with the Cayleyan transformation of E_6, the chord of contact being λ_x.

Similarly the Hessian transformation of E_2, E_5, E_6 have double contact with E_4, E_7, E_7 respectively, and many other such relations are easily inferred. Thus the Cayleyan and Hessian transformations of E_3 have double contact. Similarly the Cayleyan and Hessian transformations of E_6.

180. The discriminant with respect to k of the equation (126), is

$$E_1E_3 - E_2^2 = 0;$$

or, substituting the values E_1, E_2, E_3, the discriminant is

$$PR + Q^2 = 0. \tag{143}$$

Now, if the point x be fixed, this will be the equation of the four points of intersection of the polar conics of x with respect to the cubic; but if x vary, and the line co-ordinates fixed, the equation will be the envelope of the Hessian transformation of λ_x with respect to all the cubics passing through the nine points of inflection of the fundamental cubic.

181. By the theory of Conic Sections we see that R is the reciprocal of P with respect to Q. Hence the three conics P, Q, R have a common self-conjugate triangle, and therefore the nine conics E_1, E_2 . . . E_9 have

a self-conjugate triangle. The equation of the sides of this triangle will be the Jacobian of P, Q, R. This Jacobian will be found after an easy calculation to be the sum of three determinants, viz.:

$$2\lambda_1\lambda_2\lambda_3 \begin{vmatrix} 1, & 1, & 1, \\ \lambda_1 x_1, & \lambda_2 x_2, & \lambda_3 x_3, \\ \lambda_1^2 x_1^2, & \lambda_2^2 x_2^2, & \lambda_3^2 x_3^2, \end{vmatrix}$$

$$+ 2 \begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3, \\ x_1^2, & x_2^2, & x_3^2, \\ \lambda_1^5 x_1, & \lambda_2^5 x_2, & \lambda_3^5 x_3, \end{vmatrix}$$

$$+ \lambda_x \begin{vmatrix} x_1^2, & x_2^2, & x_3^2, \\ \lambda_1, & \lambda_2, & \lambda_3, \\ \lambda_1^4, & \lambda_2^4, & \lambda_3^4. \end{vmatrix} \qquad (144)$$

182. The condition that the point λ_x should subtend the conics P, Q, R in involution is

$$(x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3) = 0;$$

but this is the equation of the nine harmonic polars (see Art. 71). Hence we have the followingt heorem: *If from any point in any of the nine harmonic polars we draw three pairs of tangents to P, Q, R, these tangents form a pencil in involution.*

Section II.—Discriminants.

183. If a, b, c be any multiples, and if we regard $aP + bQ + cR = 0$ as a conic in point co-ordinates, we find its discriminant to be

$$\begin{aligned}&(16a^3 - 12abc - 4b^3 - c^3)(\lambda_1\lambda_2\lambda_3)^2\\ &+ (bc^2 - 4ab^2 - 8a^2c)(\lambda_1\lambda_2\lambda_3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\\ &+ ac^2(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)^2. \qquad (145)\end{aligned}$$

This equated to zero denotes the product of two curves of the third class, touching the cuspidal tangents of the Cayleyan. By means of the formula (145), we get the discriminants of the conics E_1 . . . . E_9. The following Table contains these contravariants :—

Disc. of $E_1 \equiv$ square of Cayleyan or K^2.

,, $E_2 \equiv K\{(54m^6 + (1 + 2m^3)^3)\lambda_1\lambda_2\lambda_3 - 9m^4(1 + 2m^3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\}$.

,, $E_3 \equiv \{(54m^6 + (1 + 2m^3)^3)\lambda_1\lambda_2\lambda_3 - 9m^4(1 + 2m^3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\}^2$.

,, $E_4 \equiv K^2$.

,, $$\begin{aligned}E_5 \equiv\ &(152m^3 - 336m^6 + 176m^9 - 32m^{12})(\lambda_1\lambda_2\lambda_3)^2\\ &- (2m(1 + 18m^3 - 48m^6 + 56m^9)(\lambda_1\lambda_2\lambda_3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\\ &+ m^2(1 - 12m^6 - 16m^9)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)^2.\end{aligned}$$

,, $E_6 \equiv \{(108m^3 + (1 - 4m^3)^3)\lambda_1\lambda_2\lambda_3 - 3m(1 - 4m^3)^2(\lambda_1{}^3\lambda_2{}^3\lambda_3{}^3)\}^2$.

,, $E_7 \equiv K\{(108m^3 + (1 - 4m^3)^3)\lambda_1\lambda_2\lambda_3 - 3m(1 - 4m^3)^2(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\}$.

,, $$\begin{aligned}E_8 \equiv\ &(1 - 88m^3 + 2168m^6 + 14400m^9 - 19116m^{12} + 8704m^{15} + 512m^{18})(\lambda_1\lambda_2\lambda_3)^2\\ &- 2m(1 - 70m^3 - 912m^6 + 2352m^9 - 4416m^{12} - 1512m^{15})(\lambda_1\lambda_2\lambda_3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\\ &+ m^2(10 + 8m^3)(1 + 16m^6 + 64m^{12})(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)^2.\end{aligned}$$

,, $$\begin{aligned}E_9 \equiv\ &(1 - 34m^3 - 100m^6 - 1800m^9 + 3760m^{12} + 1792m^{15} + 512m^{18})(\lambda_1\lambda_2\lambda_3)^2\\ &+ m(1 - 55m^3 + 348m^6 - 1936m^9 - 1904m^{12} - 288m^{15})(\lambda_1\lambda_2\lambda_3)(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)\\ &m^2(25m^3 - 210m^6 - 384m^9 - 160m^{12})(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)^2.\end{aligned}$$

From the foregoing Table we see that the discriminant of E_2 is a mean proportional between the discriminants of E_1 and E_3. 2°. The discriminant of E_3 is the square of the Cayleyan we should find if the Hessian were the fundamental cubic. 3°. The discriminant of E_1 is the same as the discriminant of E_4. 4°. The discriminants of E_1, E_3, E_4, E_6 are perfect squares. 5°. The discriminant of E_7 is a mean proportional between the discriminants of E_4 and E_6. Hence it follows, that if E_1 denotes a pair of lines, E_2, E_4, E_7 will each denote a pair of lines. Again, if E_3 denotes a pair of lines, so will E_2; and if E_6 be the product of two lines, E_7 will be the product of two lines.

184. If in the three conics P, Q, R we interchange point and line co-ordinates, P remains unaltered, but Q and R are interchanged. Hence $aP + bQ + cR$ will be changed into $aP + cQ + bR$, and we have the following theorem:—From the discriminant of the Zwischenform $aP + bQ + cR$, considered as a conic in point co-ordinates, we get its discriminant, considered as a conic in line co-ordinates, by changing λ_1, λ_2, λ_3 into x_1, x_2, x_3, and interchanging b and c. Hence the discriminant is

$$\begin{aligned}&(16a^3 - 12abc - b^3 - 4c^3)(x_1^2 x_2^2 x_3^2)\\ &- (8a^2b + 4ac^2 - b^2c)(x_1x_2x_3)(x_1^3 + x_2^3 + x_3^3) + ab^2(x_1^3 + x_2^3 + x_3^3)^2. \qquad (146)\end{aligned}$$

This equation is the product of two cubics passing through the nine points of inflection. Now, since

$$x_1x_2x_3 = \frac{m^2U + H}{1 + 8m^3},$$

$$x_1^3 + x_2^3 + x_3^3 = \frac{(1 + 2m^3)U - 6mH}{(1 + 8m^3)}.$$

Substituting these values, we get the discriminant expressed in terms of U and H. Thus the co-efficient

$$\begin{aligned}&\text{of } U^2 \text{ is } m^4(16a^3 - 12abc - b^3 - 4c^3) - m^2(1 + 2m^3)(8a^2b + 4ac^2 - b^2c) + (1 + 2m^3)^2 ab^2;\\ &\text{,, } UH \text{ is } 2m^2(16a^3 - 12abc - b^3 - 4c^3) - (1 - 4m^3)(8a^2b + 4ac^2 - b^2c) - 12m(1 + 2m^3)ab^2;\\ &\text{,, } H^2 \text{ is } (16a^3 - 12abc - b^3 - 4c^3) + 6m(8a^2b + 4ac^2 - b^2c) + 36m^2ab^2. \qquad (147)\end{aligned}$$

185. The following Table contains the discriminants of $E_1 \ldots . E_9$ considered as conics in line co-ordinates:—

Disc. of $E_1 \equiv$ square of Hessian, or H^2.

$$,, \quad E_2 \equiv (1 - 7m^3 - 456m^6 - 760m^9 - 32m^{12} + 768m^{15}) U^2$$
$$- 16m^2 (1 + 14m^3 - 75m^6 - 4m^9 + 64m^{12}) UH$$
$$- (1 - 4m^3 - 1128m^6 - 544m^9 - 128m^{12}) H^2 .$$

$,,\quad E_3 \equiv$ square of Hessian of Hessian, viz.:

$$\{(4m^2(1+2m^3)^3 - 108m^8) U - \{(1+2m^3)^3 - 18m^3(1+2m^3)^2 - 108m^6\} H\}^2.$$

$$,, \quad E_4 \equiv H^2 .$$

$$,, \quad E_5 \equiv 16m^7(1+2m^3)^3(1-m^3) U^2 + m^2(1-m^3)(1+11m^3+24m^6-112m^9-32m^{12}) UH$$
$$+ (1 + 12m^3 - 184m^6 - 1312m^9 - 192m^{12} - 128m^{15}) H^2 .$$

$,,\quad E_6 \equiv$ square of

$$\{m^2(1 - 66m^3 + 48m^6 - 64m^9) + 9(1 - 2m^3 - 8m^6)\} U$$
$$+ (1 - 120m^3 + 264m^6 - 64m^9) H .$$

$$,, \quad E_7 \equiv 8m^6 (3 + 2m^3 + 4m^6) U^2$$
$$+ 4m^2 (1 + 14m^3 + 21m^6 - 137m^9 + 48m^{12}) UH$$
$$+ (1 - 4m^3 - 288m^6 - 1408m^9 - 608m^{12}) H^2 .$$

$$,, \quad E_8 \equiv 4m^2 (1 - 6m^3 - 255m^6 - 884m^9 + 1790m^{12} - 8214m^{15} - 512m^{18}) UH$$
$$- (1 + 8m^3)^2 (1 - 8m^3 + 288m^6 + 416m^9 + 32m^{12}) H^2 .$$

$$,, \quad E_9 \equiv m(1 - 25m^3 - 164m^6 + 4096m^9 + 25876m^{12} - 8214m^{15} - 17408m^{18} - 4096m^{21}) U^2$$
$$- 4m^2(8 - 48m^3 - 2015m^6 - 7077m^9 + 16339m^{12} - 3072m^{15} - 4021m^{18}) UH$$
$$- (1 - 180m^3 - 3520m^6 - 960m^9 + 66998m^{12} - 23552m^{15} + 20480m^{18}) H^2 .$$

186. We have given in equation 145, Art. 183, the discriminant of $aP + bQ + cR$ considered as a conic in point co-ordinates; and if we perform on this discriminant the operation $\left(a' \frac{d}{da} + b' \frac{d}{db} + c' \frac{d}{dc}\right)$, we have

the condition that $a'P + b'Q + c'R$ may be conjugate (see Art. 1.) to $aP + bQ + cR$, viz., this is

$$\begin{aligned}&\{(48a^2 - 12bc)a' - (12ac + 12b^2)b' - (3c^2 + 12ab)c'\}(\lambda_1\lambda_2\lambda_3)^2 \\ &- \{(4b^2 + 16ac)a' + (8ab - c^2)b' + (8a^2 + 2bc)c'\}(\lambda_1\lambda_2\lambda_3)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) \\ &+ (2acc' + a'c^2)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^2, \qquad (148)\end{aligned}$$

which denotes the product of the equations of two curves of the third class, each touching the nine cuspidal tangents of the Cayleyan.

187. The following notation will be found convenient:—If we have two conics, S, S' of the system $aP + bQ + cR$, and that we want to state the condition that the second is conjugate to the first in point co-ordinates, we shall express it by including them in brackets with three points, thus $(SS'\therefore)$. The corresponding relation in line co-ordinates will be written $(SS'///)$.

188. *If S', S'', S''' be three conics connected by a linear relation, then the contravariants $(SS'\therefore)$, $(SS''\therefore)$, $(SS'''\therefore)$ are connected by a linear relation.*

Demonstration.—Let $S' = a'P + b'Q + c'R$; $S'' = a''P + b''Q + c''R$; $S''' = a'''P + b'''Q + c'''R$; then, if S', S'', S''' be connected by a linear relation, we have the determinant $(a'b''c''') = 0$. Now, let the discriminant of $aP + bQ + cR$ be denoted for shortness by

$$A(\lambda_1\lambda_2\lambda_3)^2 + B(\lambda_1\lambda_2\lambda_3)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + C(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^2;$$

then we have (see Art. 186)—

$$\begin{aligned}\left(SS'\therefore\right) &= \left(a'\frac{dA}{da} + b'\frac{dA}{db} + c'\frac{dA}{dc}\right)\left(\lambda_1\lambda_2\lambda_3\right)^2 \\ &+ \left(a'\frac{dB}{da} + b'\frac{dB}{db} + c'\frac{dB}{dc}\right)\left(\lambda_1\lambda_2\lambda_3\right)\left(\lambda_1^3 + \lambda_2^3 + \lambda_3^3\right) \\ &+ \left(a'\frac{dC}{da} + b'\frac{dC}{db} + c'\frac{dC}{dc}\right)\left(\lambda_1^3 + \lambda_2^3 + \lambda_3^3\right)^2.\end{aligned}$$

With similar expressions for $(SS''\therefore)$, $(SS'''\therefore)$.

Hence the condition that $(SS'.\because.)$, $(SS''.\because.)$, $(SS'''.\because.)$ may be connected by a linear relation is the vanishing of the product of the determinants

$$\begin{vmatrix} a', & b', & c', \\ a'', & b'', & c'', \\ a''', & b''', & c''', \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \frac{dA}{da}, & \frac{dA}{db}, & \frac{dA}{dc}, \\ \frac{dB}{da}, & \frac{dB}{db}, & \frac{dB}{dc}, \\ \frac{dC}{da}, & \frac{dC}{db}, & \frac{dC}{dc}. \end{vmatrix}$$

Hence the proposition is proved.

Cor.—If S', S'', S''' be connected by a linear relation, the three contravariants $(SS'.\because.)$, $(SS''.\because.)$, $(SS'''.\because.)$ touch each cuspidal tangent in six points in involution. Hence we infer that the several systems

$$(EE_1.\because.),\ (EE_3.\because.),\ (EE_4.\because.);\qquad (EE_3.\because.),\ (EE_6.\because.),\ (EE_4.\because.);$$
$$(EE_1.\because.),\ (EE_6.\because.),\ (EE_4.\because.);\qquad (EE_1.\because.),\ (EE_6.\because.),\ (EE_3.\because.);$$

&c., &c.,

where E is any conic of the form $aP + bQ + cR$, are such that the three pairs of curves in each system touch each cuspidal tangent of the Cayleyan in six points in involution.

189. From recent Articles it is plain that if we take E_1, E_4, E_7, which are the Hessian, the Cayleyan, and the Hesso-Cayleyan transformations of λ_x, as our fundamental Zwischenforms, the six remaining conics, namely, E_2, E_3, E_5, E_6, E_8, E_9 can be expressed linearly in terms of them. E_1, E_4, E_7 are the simplest of the entire series, both geometrically and analytically. Clebsch takes E_1, E_2, E_3 as his fundamental forms. They are his Θ, H, K (see *Vorlesungen über Geometrie*). There is scarcely, however, anything in common in his investigations and ours. The

following Table contains the values of the contravariants ($SS'.\because$) for the six pairs got by the permutations, two by two, of E_1, E_4, E_7:—

$$(E_1E_4.\because) \equiv 36m(\lambda_1\lambda_2\lambda_3)^2 + 12m^2\lambda_1\lambda_2\lambda_3(\lambda_1^3+\lambda_2^3+\lambda_3^3) + (\lambda_1^3+\lambda_2^3+\lambda_3^3)^2;$$

$$(E_1E_7.\because) \equiv \{3\lambda_1\lambda_2\lambda_3 - m(\lambda_1^3+\lambda_2^3+\lambda_3^3)\} \times \text{Cayleyan} = K\Psi \text{ (see Art. 63)};$$

$$(E_4E_1.\because) \equiv \{18m^2(\lambda_1\lambda_2\lambda_3) + (1+2m^3)(\lambda_1^3+\lambda_2^3+\lambda_3^3)\} \times \text{Cayleyan};$$

$$\begin{aligned}(E_4E_7.\because) \equiv\ & 6m^2(5+4m^3)(1-4m^3)(\lambda_1\lambda_2\lambda_3)^2 \\ & + (1-44m^3+16m^6)\lambda_1\lambda_2\lambda_3(\lambda_1^3+\lambda_2^3+\lambda_3^3) \\ & + (1-10m^3)(\lambda_1^3+\lambda_2^3+\lambda_3^3)^2;\end{aligned}$$

$$(E_7E_1.\because) \equiv \{3(1-44m^3+16m^6)\lambda_1\lambda_2\lambda_3 - m(5-16m^3-16m^6)\lambda_1^3+\lambda_2^3+\lambda_3^3\}\times\text{Cayleyan};$$

$$\begin{aligned}(E_7E_4.\because) =\ & 12m(1-84m^3-96m^6-64m^9)(\lambda_1\lambda_2\lambda_3)^2 \\ & + 12m^2(7-56m^3-32m^6)(\lambda_1\lambda_2\lambda_3)(\lambda_1^3+\lambda_2^3+\lambda_3^3) \\ & - (1-4m^3)(1-28m^3)\lambda_1^3+\lambda_2^3+\lambda_3^3)^2.\end{aligned}$$

Since the total number of permutations of nine things, taken two at a time, is 72, it follows that instead of the foregoing six contravariants we should have 72 if we completed our Table. Some of the contravariants given above are not new: thus $18m^2\lambda_1\lambda_2\lambda_3 + (1+2m^3)\lambda_1^3+\lambda_2^3+\lambda_3^3$ is given in Cayley's Seventh Memoir on Quantics (see *Philosophical Transactions*, vol. 151).

190. If in the equation of Art. 186 we interchange b and c, and b' and c', and change $\lambda_1 . \lambda_2 . \lambda_3$ into x_1, x_2, x_3, we have the condition that $a'P + b'Q + c'R$ may be conjugate to $aP + bQ + cR$, both being considered as conics in line co-ordinates. Hence, the required condition is—

$$\begin{aligned}&\{(48a^2-12bc)a' - (3b^2+12ac)b' - (12ab+12c^2)c'\}(x_1x_2x_3)^2 \\ &- \{(16ab+4c^2)a' + (8a^2-2bc)b' + 8ac - b^2)c')\}(x_1x_2x_3)(x_1^3+x_2^3+x_3^3) \\ &+ \{2abb' + a'b^2\}(x_1^3+x_2^3+x_3^3)^2 = 0;\end{aligned} \tag{149}$$

and this denotes the product of two cubics, each passing through the nine points of inflection of the fundamental cubic. If the coefficients in

equation (147), Art. 184, be denoted by A, B, C; then equation (149) may be written as follows:—

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\right)(AU^2 + BUH + CH^2 = 0). \qquad (150)$$

191. It may be proved, as in Art. 188, if three conics S', S'', S''' be connected by a linear relation, that the covariants $(SS'///)$, $(SS''///)$, $(SS'''///)$ are connected by a linear relation (where S is any conic of the system $aP + bQ + cR$). *Hence it follows that the three pairs of tangents to these three covariants, at any of the nine points of inflection, will form a pencil in involution.* The following Table contains twelve triads possessing this property :—

$$\begin{array}{llllll}
(EE_1///), & (EE_3///), & (EE_4///); & (EE_3///), & (EE_6///), & (EE_4///); \\
(EE_1///), & (EE_6///), & (EE_4///); & (EE_1///), & (EE_6///), & (EE_3///); \\
(EE_1///), & (EE_2///), & (EE_7///); & (EE_1///), & (EE_7///), & (EE_9///); \\
(EE_2///), & (EE_7///), & (EE_9///); & (EE_1///), & (EE_2///), & (EE_9///); \\
(EE_4///), & (EE_5///), & (EE_7///); & (EE_4///), & (EE_5///), & (EE_8///); \\
(EE_4///), & (EE_7///), & (EE_8///); & (EE_5///), & (EE_7///), & (EE_8///).
\end{array}$$

192. The following Table, which corresponds to that of Art. 189, has the covariant expressed in terms of U and H:—

$$\begin{aligned}
(E_4E_1///) &\equiv 4S^2U^2 + 3TUH - 36SH^2, \\
(E_7E_1///) &\equiv (7m^4 - 10m^7 + 252m^{10} - 24m^{13} + 16m^{16})U^2 \\
&\quad + 4m(2 - 31m^3 + 32m^6 + 24m^{10})SUH \\
&\quad + 3(1 - 16m^3 + 140m^6 + 232m^9 + 48m^{12})H^2. \\
(E_1E_4///) &\equiv TUH - 24SH^2, \\
(E_1E_7///) &\equiv 6m^4(1 + 2m^3)(1 - 4m^3)U^2 \\
&\quad + 2m^2(1 + 2m^3)(7 - 14m^3 + 16m^6)UH \\
&\quad + 3(1 - 24m^3 - 120m^6 - 64m^9)H^2. \\
(E_4E_7///) &\equiv UH. \qquad \text{(Compare } (E_1E_7 \therefore).) \\
(E_7E_4///) &\equiv \{(1 + 2m^3)(1 - 4m^3)U - 12SH\}H.
\end{aligned}$$

193. The condition that the conics $aP + bQ + cR$, $a'P + b'Q + c'R$ may be cut harmonically by the line λ_x is—

$$\begin{aligned}
&(48aa' - 6bc' - 6b'c)(\lambda_1\lambda_2\lambda_3)^2 \\
&+ 4(bb' - ac' - a'c)(\lambda_1\lambda_2\lambda_3)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) \\
&+ 4(cc' + 2ab' + 2a'b)(\lambda_2^3\lambda_3^3 + \lambda_3^3\lambda_1^3 + \lambda_1^3\lambda_2^3) \\
&- cc'(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^2 = 0: \qquad (151)
\end{aligned}$$

hence, the condition that E_1 and E_4 may be cut harmonically by λ_x is—

$$\begin{aligned}
&12m^2(\lambda_1\lambda_2\lambda_3)^2 - (1 - 4m^3)(\lambda_1\lambda_2\lambda_3)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) \\
&+ S(\lambda_1^3\lambda_2^3 + \lambda_2^3\lambda_3^3 + \lambda_3^3\lambda_1^3) - m(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^2.
\end{aligned}$$

194. If $aP + bQ + cR$, $a'P + b'Q + c'R$ be considered as conics in point co-ordinates, we have, from Art. 193, by interchanging b and c, b' and c', and changing λ_1, λ_2, λ_3 into x_1, x_2, x_3, the condition that the pencil of tangents from the point λ_x to these conics will be a harmonic pencil, viz., this is

$$\begin{aligned}
&(48aa' - 6bc' - 6b'c(x_1x_2x_3)^2 \\
&+ (cc' - ab' - a'b)(x_1x_2x_3)(x_1^3 + x_2^3 + x_3^3) \\
&+ 4(bb' + 2ac' + 2a'c)(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) \\
&- bb'(x_1^3 + x_2^3 + x_3^3)^2 = 0. \qquad (152)
\end{aligned}$$

Now, substituting for $x_1x_2x_3$ its value

$$\frac{m^2U + H}{1 + 8m^3},$$

" $x_1^3 + x_2^3 + x_3^3$ its value

$$\frac{(1 + 2m^3)U - 6mH}{1 + 8m^3},$$

" $x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3$ its value

$$\frac{m^3(2 + m^3)U^2 - m(1 + 2m^3)UH + 3m^2H - \Theta}{(1 + 8m^3)^2},$$

we have our equation (152) expressed in terms of the fundamental covariants U, H, Θ (see Salmon's *Higher Curves*, page 194). In this way we get

$$
\begin{aligned}
&\{m^4(48aa' - 6bc' - 6b'c) + 4m^2(1 + 2m^3)(cc' - a'b - ab') \\
&+ 4m^3(2 + m^3)(bb' + 2ac' + 2a'c) - (1 + 2m^3)^2 bb'\}\, U^2 \\
&+ \{2m^2(48aa' - 6bc' - 6b'c) + 4(1 - 4m^3)(cc' - a'b - ab') \\
&- 4m(1 + 2m^3)(bb' + 2ac' + 2a'c) + 12m(1 + 2m^3)bb'\}\, UH \\
&+ \{(48aa' - 6bc' - 6b'c) - 24m(cc' - a'b - ab') \\
&+ 12m^2(bb' + 2ac' + 2a'c) - 36m^2bb'\}\, H^2 \\
&- 4(bb' + 2ac' + 2a'c)\Theta = 0.
\end{aligned}
\tag{153}
$$

Cor. 1.—If $bb' + 2ac' + 2a'c = 0$, the equation (153) will break up into two cubics, each passing through the nine points of inflection.

Cor. 2.—If we take for our Zwischenform E_1 and E_3, we have

$$
\begin{array}{lll}
a = m^2, & b = 2m, & c = -1, \\
a' = (1 + 2m^3)^2, & b' = -12m^2(1 + 2m^3), & c' = -36m^4;
\end{array}
$$

and, substituting these values in equation (153), it reduces to

$$8(1 + 8m^3)^2\Theta.$$

Hence, omitting the numerical multiplier, it reduces to Θ, as we know it ought.

195. If we denote the locus of the point λ_x, in the equation (153), for the two conics E_s, E_t, by the harmonic of E_s and E_t, or briefly by har. (E_sE_t), the following Table contains this covariant for the three combinations formed with E_1, E_4, E_7:—

$$
\begin{aligned}
\text{Har.}\,(E_1E_4) &\equiv 3m^6U^2 - 2m(1 + 4m^3 - 8m^6)UH \\
&\quad + 3m^2H^2 - (1 + 8m^3)\Theta;
\end{aligned}
$$

$$\text{Har.}\,(E_4E_7) \equiv 4\,TSUH - 48S^2H^2 + T\Theta.$$

This curve passes through the points of intersection of Θ and H, and must

therefore touch H at the nine points which correspond to the nine points of inflection on H.

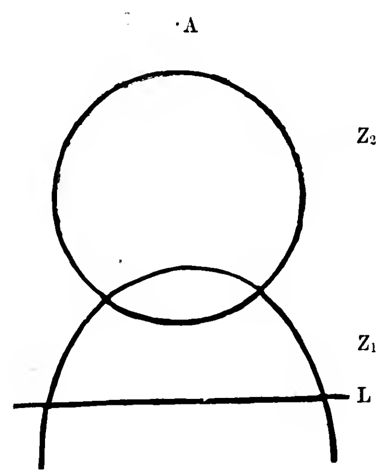
$$\text{Har.}\,(E_1E_7) \equiv 2S^2U^2 + TUH - 6SH^2.$$

This is the product of two cubics passing through the nine points of inflection.

196. Since a', b', c' enter only in the first degree in the equation (153), we infer the following theorem: *If E_r, E_s, E_t, three conics of the form $aP + bQ + cR$, be connected by a linear relation, and if E be any other conic of the same system, then the three sextics, har. (EE_r), har. (ES_s), har. (EE_t) are connected by a linear relation—in other words, they pass through the same thirty-six points.*

SECTION III.

197. In Art. 109, *Cor.* 2, we derived from the conic π a remarkable cubic. The process of that Article can be generalized as follows, thus: let Z be any Zwischenform conic, and if in Z we substitute in succession the co-ordinates of a fixed point A, and of a fixed line L, we get two conics—say Z_1, Z_2—in line and point co-ordinates respectively, thus: let



$$Z = f(x_1,\ x_2,\ x_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),$$

and let the point co-ordinates of A be y_1, y_2, y_3, and the line co-ordinates of L be μ_1, μ_2, μ_3, then we have

$$Z_1 = f(y_1,\ y_2,\ y_3,\ \lambda_1,\ \lambda_2,\ \lambda_3)$$

(this will be a conic in line co-ordinates, since the y's are constant and the λ's variables):

$$Z_2 = f(x_1,\ x_2,\ x_3,\ \mu_1,\ \mu_2,\ \mu_3),$$

which will be a conic in point co-ordinates. From the form of these

equations, it is evident that if L be a tangent to Z_1, A will be a point on Z_2, and in this case the condition is fulfilled, that Z_1 and Z_2 are mutually conjugate.

198. We shall now investigate the condition that the polar line of A with respect to Z_2 shall pass through the pole of L with respect to Z_1: obviously, the required condition is

$$\frac{dZ}{d\lambda_1}\cdot\frac{dZ}{dx_1}+\frac{dZ}{d\lambda_2}\cdot\frac{dZ}{dx_2}+\frac{dZ}{d\lambda_3}\cdot\frac{dZ}{dx_3}=0\,; \tag{154}$$

where, after the differentiation, we are to substitute the point co-ordinates of A and the line co-ordinates of L. The equation is a relation between the co-ordinates of A and L. If we suppose the co-ordinates of A fixed, while those of L vary, it will be a cubic in line co-ordinates, and L must be a tangent to it. Again, if we suppose the co-ordinates of L fixed, while the point A varies, it will be a cubic in point co-ordinates, and A must be a point on the curve: in fact, one of the two curves we thus get will be the envelope of L, and the other will be the locus of A. We shall call this *Zwischenform* the *bicubic* of Z. We give here the bicubics of P, Q, R:—

$$\sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dP}{dx_s}\right)=4P\lambda_x+48\lambda_1\lambda_2\lambda_3x_1x_2x_3\,; \tag{155}$$

$$\sum_{s=1}^{s=3}\left(\frac{dQ}{d\lambda_s}\cdot\frac{dQ}{dx_s}\right)=2R\lambda_x-2\left(\lambda_1^3+\lambda_2^3+\lambda_3^3\right)x_1x_2x_3\,; \tag{156}$$

$$\sum_{s=1}^{s=3}\left(\frac{dR}{d\lambda_s}\cdot\frac{dR}{dx_s}\right)=2Q\lambda_x-2\left(\lambda_1\lambda_2\lambda_3\right)\left(x_1^3+x_2^3+x_3^3\right). \tag{157}$$

199. If we write out the bicubic of Q, we get

$$\begin{aligned}&2\lambda_1^2\lambda_2x_2^2x_3+2\lambda_2^2\lambda_3x_3^2x_1+2\lambda_3^2\lambda_1x_1^2x_2\\&+2\lambda_1\lambda_2^2x_3x_1^2+2\lambda_2\lambda_3^2x_1x_2^2+2\lambda_3\lambda_1^2x_2x_3^2.\end{aligned} \tag{158}$$

Hence, the bicubic of Q, considered as a cubic in point co-ordinates, is the sum of two cubics, each described about the triangle of reference, and each angular point is its own third tangential with respect to each cubic. The following properties of this cubic may be mentioned:—

1°. The three lines, from the angular points to where the opposite sides meet the cubic again, are $\lambda_1 x_1 + \lambda_2 x_2$, $\lambda_2 x_2 + \lambda_3 x_3$, $\lambda_3 x_3 + \lambda_1 x_1$, respectively; and the three points, where the sides of the triangle meet the cubic again, are their intersections with the line λ_x.

2°. The conic

$$\frac{\lambda_1^2}{x_1} + \frac{\lambda_2^2}{x_2} + \frac{\lambda_3^2}{x_3} = 0$$

has triple contact with the cubic, for the three lines

$$\frac{x_1}{\lambda_1^2} + \frac{x_2}{\lambda_2^2}, \quad \frac{x_2}{\lambda_2^2} + \frac{x_3}{\lambda_3^2}, \quad \frac{x_3}{\lambda_3^2} + \frac{x_1}{\lambda_1^2},$$

are tangents to the conic and cubic at the angular points of the triangle.

200. If we write out the bicubic of R, we find its tangents at the angular points $\lambda_1 x_1 + \lambda_2 x_2$, $\lambda_2 x_2 + \lambda_3 x_3$, $\lambda_3 x_3 + \lambda_1 x_1$, while the lines drawn from the angular points to where the curve meets the sides of the triangle in the third intersections are—

$$\frac{x_1}{\lambda_1^2} + \frac{x_2}{\lambda_2^2}, \quad \frac{x_2}{\lambda_2^2} + \frac{x_3}{\lambda_3^2}, \quad \frac{x_3}{\lambda_3^2} + \frac{x_1}{\lambda_1^2},$$

respectively. These are the lines in the last Article, but in a reversed order.

201. The Hessians of the bicubics of Q and R can be written in terms of the tangents to these cubics at the angular points of the triangle of reference, thus denoting the tangents to the former, namely:—

$$\frac{x_2}{\lambda_2^2} + \frac{x_3}{\lambda_3^2}, \quad \frac{x_3}{\lambda_3^2} + \frac{x_1}{\lambda_1^2}, \quad \frac{x_1}{\lambda_1^2} + \frac{x_2}{\lambda_2^2},$$

by L, M, N respectively, and the tangents to the latter by L', M', N'. The Hessians of the two bicubics will be

$$(\lambda_1\lambda_2\lambda_3)^3 LMN + 2L'M'N' - \lambda_1^3 LL'^2 - \lambda_2^3 MM'^2 - \lambda_3^3 NN'^2 = 0; \qquad (159)$$

$$2(\lambda_1\lambda_2\lambda_3)^3 LMN + L'M'N' - (\lambda_1\lambda_2\lambda_3)^2(L^2L' + M^2M' + N^2N') = 0. \qquad (160)$$

202. In a similar way we may define the bicubic of two conics P and Q:—

$$\sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dQ}{dx_s}\right) = 0; \qquad (161)$$

the interpretation of which is obvious. After an easy calculation we get

$$\sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dQ}{dx_s} + \frac{dP}{dx_s}\cdot\frac{dQ}{d\lambda_s}\right)$$

$$= 4\lambda_1\lambda_2\lambda_3(x_1^3 + x_2^3 + x_3^3) - 4Q\lambda_x; \qquad (162)$$

$$\sum_{s=1}^{s=3}\left(\frac{dR}{d\lambda_s}\cdot\frac{dP}{dx_s} + \frac{dR}{dx_s}\cdot\frac{dP}{d\lambda_s}\right)$$

$$= 4(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)(x_1x_2x_3) - 4R\lambda_x; \qquad (163)$$

$$\sum_{s=1}^{s=3}\left(\frac{dQ}{d\lambda_s}\cdot\frac{dR}{dx_s} + \frac{dQ}{dx_s}\cdot\frac{dR}{d\lambda_s}\right)$$

$$= (\lambda_1^3+\lambda_2^3+\lambda_3^3)(x_1^3+x_2^3+x_3^3) + 6\lambda_1\lambda_2\lambda_3\, x_1x_2x_3 - P\lambda_x. \qquad (164)$$

203. From the results of Articles 198, 202, we infer the two following identities:—

$$\sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dQ}{dx_s} + \frac{dP}{dx_s}\cdot\frac{dQ}{d\lambda_s} + 2\frac{dR}{d\lambda_s}\cdot\frac{dR}{dx_s}\right) = 0; \qquad (165)$$

$$\sum_{s=1}^{s=3}\left(\frac{dR}{d\lambda_s}\cdot\frac{dP}{dx_s} + \frac{dR}{dx_s}\cdot\frac{dP}{d\lambda_s} + 2\frac{dQ}{d\lambda_s}\cdot\frac{dQ}{dx_s}\right) = 0. \qquad (166)$$

204. The bicubic of $aP + bQ + cR$ can be easily calculated from the results of recent Articles: thus we get it equal

$$a^2 \sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dP}{dx_s}\right) + b^2 \sum_{s=1}^{s=3}\left(\frac{dQ}{d\lambda_s}\cdot\frac{dQ}{dx_s}\right)$$

$$+ c^2 \sum_{s=1}^{s=3}\left(\frac{dR}{d\lambda_s}\cdot\frac{dR}{dx_s}\right) + ab \sum_{s=1}^{s=3}\left(\frac{dP}{d\lambda_s}\cdot\frac{dQ}{dx_s} + \frac{dP}{dx_s}\cdot\frac{dQ}{d\lambda_s}\right)$$

$$+ bc \sum_{s=1}^{s=3}\left(\frac{dQ}{d\lambda_s}\cdot\frac{dR}{dx_s} + \frac{dQ}{dx_s}\cdot\frac{dR}{d\lambda_s}\right)$$

$$+ ca \sum_{s=1}^{s=3}\left(\frac{dR}{d\lambda_s}\cdot\frac{dP}{dx_s} + \frac{dR}{dx_s}\cdot\frac{dP}{d\lambda_s}\right). \qquad (167)$$

Hence, substituting the values, we get the bicubic of $aP + bQ + cR$

$$= \{(4a^2 - bc)P + (2c^2 - 4ab)\,Q + (2b^2 - 4ac)R\}\lambda_x$$
$$+ \{bc(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + (4ab - 2c^2)\lambda_1\lambda_2\lambda_3\}(x_1^3 + x_2^3 + x_3^3)$$
$$+ \{(48a^2 + 6bc)\lambda_1\lambda_2\lambda_3 + (4ac - 2b^2)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)\}x_1x_2x_3. \qquad (168)$$

This formula is perfectly symmetrical in point and line co-ordinates; so that if we interchange b and c, and change λ_1, λ_2, λ_3 into x_1, x_2, x_3, and *vice versâ*, the formula remains unaltered.

205. The following Table contains the bicubics for the nine Zwischenforms or biconics $E_1 \ldots . E_9$. Thus the bicubic of

E_1 is $E_2\lambda_x - KU$,

E_2 ,, $TE_1\lambda_x - \{3m^2(1-4m^3)(\lambda_1^3+\lambda_2^3+\lambda_3^3) + 2mT\lambda_1\lambda_2\lambda_3\}x_1^3+x_2^3+x_3^3$
$+ \{(54m^2(1-4m^3) - 12m^2T)\lambda_1\lambda_2\lambda_3 - T(\lambda_1^3+\lambda_2^3+\lambda_3^3)\}\,x_1x_2x_3.$

In this equation T is the sextic invariant of the cubic.

E_3 is $[\{(1+2m^3)^4 - 108m^6(1+2m^3)\}P + 12m^2\{54m^6+(1+2m^3)^3\}Q$
$+ 108m^4(1+2m^3)^2R]\lambda_x$
$+ \{108m^6(1+2m^3)(\lambda_1^3+\lambda_2^3+\lambda_3^3)$
$- 12m^2(1+2m^3)(1+6m^3+66m^6+8m^9)\}(x_1^3+x_2^3+x_3^3)$
$+ \{12(1+2m^3)(1+6m^3+66m^6+8m^9)\lambda_1\lambda_2\lambda_3$
$- 108m^4(1+2m^3)^2(\lambda_1^3+\lambda_2^3+\lambda_3^3)\}x_1x_2x_3.$

E_4 ,, $E_5\lambda_x - 4\Psi H.$

E_5 ,, $TE_4\lambda_x + 4m^2\{(1+2m^3)\lambda_1^3+\lambda_2^3+\lambda_3^3) + T\lambda_1\lambda_2\lambda_3\}x_1^3+x_2^3+x_3^3$
$+ 4\{(1-2m^3+28m^6)\lambda_1\lambda_2\lambda_3 + mT(\lambda_1^3+\lambda_2^3+\lambda_3^3)\}x_1x_2x_3.$

E_6 ,, $[\{(1-4m^3)^4 + 108m^3(1-4m^3)\}P + 108m^2(1-4m^3)^2Q$
$- 12m\{(1-4m^3)^3 - 54m^3\}R]\lambda_x$
$- 108m^2(1-4m^3)K(x_1^3+x_2^3+x_3^3)$
$+ 12\{(1-4m^3)^3 - 54m^3\}Kx_1x_2x_3.$

E_7 ,, $-2T(mP - Q + 2m^2R)\lambda_x$
$+ \{2m(1-2m^3-8m^6)(\lambda_1^3+\lambda_2^3+\lambda_3^3) - T\lambda_1\lambda_2\lambda_3\}(x_1^3+x_2^3+x_3^3)$
$+ \{12m(1+34m^3-8m^6)\lambda_1\lambda_2\lambda_3 + 4m^2T\}x_1x_2x_3.$

E_8 ,, $[m(1-60m^3-72m^6-112m^9)P$
$- (1+20m^3-168m^6-160m^9+32m^{12})Q$
$- 2m^2(7-36m^3+240m^6+32m^9\ R]\lambda_x$
$- \{m(1-10m^3+8m^6-80m^9)(\lambda_1^3+\lambda_2^3+\lambda_3^3)$
$- (1+20m^3-168m^6-160m^9+32m^{12})\lambda_1\lambda_2\lambda_3\}(x_1^3+x_2^3+x_3^3)$
$- \{6m(1+2m^3)(1+88m^3-8m^6)\lambda_1\lambda_2\lambda_3$
$- 2m^2(7-36m^3+240m^6+32m^9)(\lambda_1^3+\lambda_2^3+\lambda_3^3)\}x_1x_2x_3.$

$$E_9 \text{ is } \{m^2(7 - 36m^3 + 240m^6 + 32m^9) P$$
$$- 2m(1 - 60m^2 - 72m^6 - 112m^9) Q$$
$$+ (1 + 20m^3 - 184m^6 + 16m^9 - 96m^{12}) R\} \lambda_x$$
$$- \{m^2(5 + 4m^3)(1 + 8m^6)(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)$$
$$- 2m(1 - 60m^3 - 72m^6 - 112m^9)\lambda_1\lambda_2\lambda_3\} (x_1^3 + x_2^3 + x_3^3)$$
$$- \{6m^2(1 - 4m^3)(1 + 88m^3 + 8m^6)\lambda_1\lambda_2\lambda_3$$
$$+ (1 + 20m^3 - 184m^6 + 16m^9 - 96m^{12})(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)\} x_1x_2x_3.$$

206. The condition that the line a_x should be cut in involution by the conics P, Q, R, is

$$a_1^3\lambda_1^3(\lambda_2^3 - \lambda_3^3) + a_2^3\lambda_2^3(\lambda_3^3 - \lambda_1^3 + a_3^3\lambda_3^3(\lambda_1^3 - \lambda_2^3)$$
$$+ 2a_1a_2\lambda_1\lambda_2(a_1\lambda_2 - a_2\lambda_1)(\lambda_1^3 - \lambda_2^3)$$
$$+ 2a_2a_3\lambda_2\lambda_3(a_2\lambda_3 - a_3\lambda_2)(\lambda_2^3 - \lambda_3^3)$$
$$+ 2a_3a_1\lambda_3\lambda_1(a_3\lambda_1 - a_1\lambda_3)(\lambda_3^3 - \lambda_1^3) = 0. \quad (169)$$

Hence, the condition that λ_x should be cut in involution is

$$(\lambda_1^3 - \lambda_2^3)(\lambda_2^3 - \lambda_3^3)(\lambda_3^3 - \lambda_1^3) = 0; \quad (170)$$

but this is the equation of the nine harmonic poles of the curve Ψ. Hence we have the following theorem: *Any line passing through one of the nine harmonic poles of the curve Ψ is cut in involution by the three conics P, Q, R.*

This theorem belongs to Chapter VI., Section 1.

CHAPTER VII.

SECTION I.—CORRESPONDING POINTS.

207. If x_1, x_2, x_3 be the co-ordinates of any point on the Hessian, substituting them in its equation, we get the determinant

$$\begin{vmatrix} x_1, & mx_3, & mx_2 \\ mx_3, & x_2, & mx_1 \\ mx_2, & mx_1, & x_3 \end{vmatrix} = 0.$$

As we shall have much use to make of the minors of this determinant, we give their values and fundamental property here, thus:

$$\begin{aligned} A_{11} &= x_2x_3 - m^2x_1^2\,; & A_{23} &= m^2x_2x_3 - mx_1^2\,; \\ A_{22} &= x_3x_1 - m^2x_2^2\,; & A_{31} &= m^2x_3x_1 - mx_2^2\,; \\ A_{33} &= x_1x_2 - m^2x_3^2\,; & A_{12} &= m^2x_1x_2 - mx_3^2\,; \end{aligned}$$

hence we easily get

$$A_{11} \cdot A_{22} = A_{12}^2, \qquad A_{22} \cdot A_{33} = A_{23}^2, \qquad A_{33} \cdot A_{11} = A_{31}^2 \tag{171}$$

(see also Salmon's *Algebra*, page 29).

208. If x_1, x_2, x_3 be the co-ordinates of a point on the Hessian, the co-ordinates of its corresponding point are

$$\sqrt{A_{11}}, \quad \sqrt{A_{22}}, \quad \sqrt{A_{33}},$$

or, symbolically, A_1, A_2, A_3.

Demonstration.—From Art. (163), we see that if λ_ξ be the equation of any point on the Hessian, the square of the equation of its corresponding point is the bordered Hessian or Hessian transformation of λ_ξ, which in this case is $(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3)^2$. Hence the proposition is proved. The proposition also follows from the theory of the Hessian transformation. See Art. 32.

209. We shall verify the foregoing proposition by forming according to the same rule the co-ordinates of the point which corresponds to the point A_1, A_2, A_3, viz.: these are

$$\sqrt{A_{23} - m^2 A_{11}}, \qquad \sqrt{A_{31} - m^2 A_{22}}, \qquad \sqrt{A_{12} - m^2 A_{33}};$$

or, restoring the values of the minors A_{11}, A_{12}, &c., we get

$$\sqrt{-Sx_1^2}, \qquad \sqrt{-Sx_2^2}, \qquad \sqrt{-Sx_3^2};$$

and these are proportional to x_1, x_2, x_3. Hence the proposition is proved.

210. The line joining the points (x_1, x_2, x_3), (A_1, A_2, A_3) is a tangent to the Cayleyan. We know this ought to be the case, since it is a line joining corresponding points; but a direct proof of this proposition will be also a proof of the theorem of Art. (208), and some of the results arrived at in the demonstration will be useful in future propositions.

Let the line joining the points be λ_x, then the line co-ordinates λ_1, λ_2, λ_3 are given by the system of determinants

$$\left\| \begin{matrix} x_1, & x_2, & x_3, \\ A_1, & A_2, & A_3; \end{matrix} \right\|$$

and these being substituted in the determinant which represents the Cayleyan, the proposition will be proved by forming the reciprocal determinant, and showing that it is equal to zero. From the foregoing values we get by multiplication of determinants

$$\lambda_2\lambda_3 = \left| \begin{matrix} x_1x_3 + x_1x_2, & A_1x_3 + A_2x_1, \\ x_1A_3 + x_2A_1, & A_{13} + A_{12}. \end{matrix} \right|$$

Hence, restoring the values of A_{11}, A_{12}, &c., we get

$$4m^2\lambda_2\lambda_3 = 8m^3x_1^4 - 4m^2x_2^2x_3^2 - 4mx_1^2x_2x_3.$$

In like manner

$$\lambda_1^2 = x_1(x_2^3 + x_3^3) - 4m^2x_2^2x_3^2 + 2mx_1^2x_2x_3;$$

$$\begin{aligned} \therefore \quad m^2(4m^2\lambda_2\lambda_3 - \lambda_1^2) &= 8m^5x_1^4 - m^2x_1(x_2^3 + x_3^3 - 2mx_1x_2x_3) - 8m^3x_1^2x_2x_3 \\ &= (8m^5 + m^2)x_1^4 - (1 + 8m^3)x_1^2x_2x_3 \\ &= -(1 + 8m^3)x_1^2A_{11}. \end{aligned} \tag{172}$$

And, making similar calculations for the other minors of the Cayleyan, we get the reciprocal determinant

$$= (1 + 8m^3)^3 \begin{vmatrix} x_1^2 A_{11}, & x_1 x_2 A_{12}, & x_1 x_3 A_{13} \\ x_1 x_2 A_{12}, & x_2^2 A_{22}, & x_2 x_3 A_{23} \\ x_1 x_3 A_{13}, & x_2 x_3 A_{23}, & x_3^2 A_{33} \end{vmatrix}$$

$$= (1 + 8m^3)^3 x_1^2 x_2^2 x_3^2 \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{13}, & A_{23}, & A_{33} \end{vmatrix}$$

$$= (1 + 8m^3)^3 x_1^2 x_2^2 x_3^2 H^2 = 0. \qquad (173)$$

Hence the proposition is proved.

211. We have seen, Art. 133, that when the line λ_x is a tangent to the Cayleyan, the bordered Cayleyan is a perfect square, and denotes the square of its corresponding tangent. Hence, if we denote this latter line by μ_x, we have the following equalities, viz. :—

$$\mu_1^2 = 4m^2 \lambda_2 \lambda_3 - \lambda_1^2. \qquad (174)$$

$$\mu_2^2 = 4m^2 \lambda_3 \lambda_1 - \lambda_2^2. \qquad (175)$$

$$\mu_3^2 = 4m^2 \lambda_1 \lambda_2 - \lambda_3. \qquad (176)$$

Hence, from the values of $4m^2\lambda_2\lambda_3 - \lambda_1^2$, $4m^2\lambda_3\lambda_1 - \lambda_2^2$, $4m^2\lambda_1\lambda_2 - \lambda_3^2$, got in the last Article, we have, omitting numerical multipliers,

$$\mu_1 = x_1 A_1. \qquad (177)$$

$$\mu_2 = x_2 A_2. \qquad (178)$$

$$\mu_3 = x_3 A_3. \qquad (179)$$

In these equations, A_1, A_2, A_3 are no longer umbral, but denote real quantities, viz., the co-ordinates of the point corresponding to x_1, x_2, x_3.

Cor.—From the values of μ_1^2, μ_2^2, μ_3^2, given in equations (174), (175),

(176), we get, by the property of determinants, the following identities:—

$$\mu_2\mu_3 = \lambda_2\lambda_3 + 2m\lambda_1^2. \tag{180}$$

$$\mu_3\mu_1 = \lambda_3\lambda_1 + 2m\lambda_2^2. \tag{181}$$

$$\mu_1\mu_2 = \lambda_1\lambda_2 + 2m\lambda_3^2. \tag{182}$$

212. From the equations (177)–(179) we have the following important theorem: *If a tangent L to the Cayleyan cuts the Hessian in two corresponding points x and y, and in a third point z, the line co-ordinates of the tangent L′, drawn through z conjugate to L, are, respectively,*

$$x_1y_1, \qquad x_2y_2, \qquad x_3y_3;$$

or, denoting current co-ordinates by α_1, α_2, α_3, *the equation of the line L′ is*

$$(x_1y_1)\,\alpha_1 + (x_2y_2)\,\alpha_2 + (x_3y_3)\,\alpha_3 = 0. \tag{183}$$

213. If we denote the two companion cubics to our fundamental cubic, as in Art. 80, by

$$x_1^3 + x_2^3 + x_3^3 + 6m'x_1x_2x_3,$$

$$x_1^3 + x_2^3 + x_3^3 + 6m''x_1x_2x_3,$$

or, say U', U''; then, although the three cubics, U, U', U'', have the same Hessian, the minors of these Hessians are different, and there will therefore be a Hessian transformation corresponding to each of these cubics. The substitutions in the transformations which correspond to U', U'' will be A_1', A_2', A_3'; A_1'', A_2'', A_3'', respectively, where these A's have interpretations exactly corresponding to those of A_1, A_2, A_3 (see Art. 37): hence we have, in all, three Hessian transformations.

Cor. 1.—*In like manner we have three Cayleyan transformations corresponding to one Cayleyan.*

Cor. 2.—*If the Hessian be given, there are three corresponding Cayleyans, viz., one for each fundamental cubic.*

Cor. 3.—The three Cayleyans may be written in the following symmetrical manner:—

$$(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) - (4m^2 + 2m'm'')\lambda_1\lambda_2\lambda_3. \qquad (184)$$

$$(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) - (4m'^2 + 2m''m)\lambda_1\lambda_2\lambda_3. \qquad (185)$$

$$(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) - (4m''^2 + 2m\,m')\lambda_1\lambda_2\lambda_3. \qquad (186)$$

214. *If the three Hessian transformations be applied in succession to the equation of any curve, the result will be the original curve.*

Demonstration.—The first Hessian transformation of x_1^2 is $x_2x_3 - m^2x_1^2$; and, applying the second transformation, we find

$$m'^2x_2x_3 - m'x_1^2 - m^2(x_2x_3 - m'^2x_1^2),$$

or

$$(m'^2 - m^2)x_2x_3 - (m' - m^2m'^2)x_1^2;$$

and the result of the third Hessian transformation will be—

$$(m'^2 - m^2)(m''^2x_2x_3 - m''x_1^2) - m' - m^2m'^2)(x_2x_3 - m''^2x_1^2),$$

or

$$(m'^2m''^2 - m''^2m^2 + m^2m' - m')x_2x_3$$
$$- (m'^2m'' + m'm''^2 - m''m^2 + m^2m'^2m''^2)x_1^2.$$

Now, reducing this by means of the equations (55–61), Articles 80, 81, we easily find the coefficient of x_2x_3 to be equal zero, while the coefficient of x_1^2 is

$$\frac{(m - m')(m' - m'')(m'' - m')}{2},$$

and we get corresponding results for the transformations of x_2^2, x_3^2.

In like manner it will be found that the result of the three transformations applied in succession to x_2x_3 is

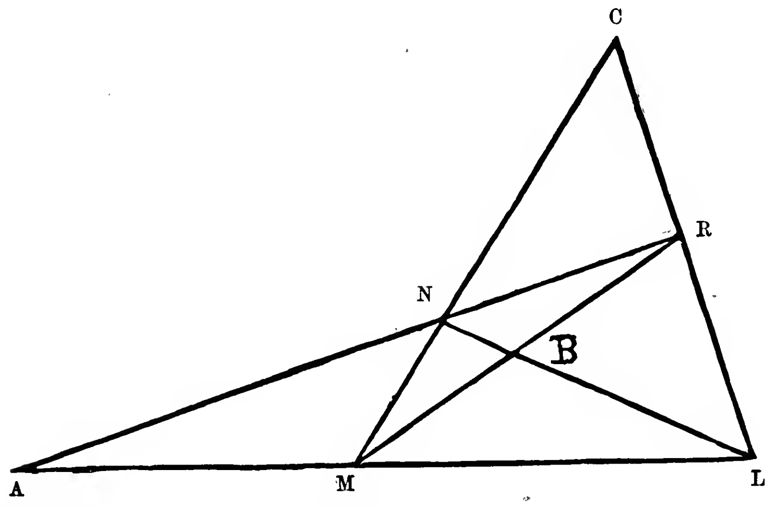
$$\frac{m - m'(m' - m'')(m'' - m)}{2}x_2x_3.$$

Now, since in applying these transformations to the equation of any curve, it is for the squares and products of the variables we make the substitutions; hence it follows that the three transformations applied in succession to the equation of any curve leaves it unaltered except by a numerical multiplier.

Cor. 1.—*In like manner, the three Cayleyan transformations applied in succession to any tangential equation leaves it unaltered except by a numerical multiplier.*

Cor. 2.—*Any two of the Hessian transformations applied in succession is equivalent to the third; and any two Cayleyan transformations applied in succession to any figure in tangential equations produces the same result as the third.*

215. Let L be a point on the Hessian, and let the three transformations of L be the points M, N, R respectively: namely, M the first, N the second, and R the third; then it follows, from the last Article, that N is the third



transformation of M, and R the first of N; or, more generally, the co-ordinates of any two points will be the same transformations of the co-ordinates of the two remaining points.

Cor.—*If the co-ordinates of the point L be* x_1, x_2, x_3, *then the co-ordinates of the three remaining points will be as follows, viz.*:—

Of M, ... A_1, A_2, A_3.

,, N, ... A_1', A_2', A_3'.

,, R, ... A_1'', A_2'', A_3''.

Definition.—*A system of four corresponding points, such as L, M, N, R, will be called a quadruple system.*

216. *The three determinants formed with the co-ordinates of any two of the four points of a quadruple system are proportional to the products of the corresponding co-ordinates of the two remaining points.*

Demonstration.—Take the two points N and R; we have the three determinants—

$$\left\| \begin{matrix} A_1', & A_2', & A_3', \\ A_1'', & A_2'', & A_3''. \end{matrix} \right\|$$

Now take the determinant got by omitting the first column, and squaring, we get, by the property of the constituents,

$$A_{22}' \,.\, A_{33}'' + A_{33}' \,.\, A_{22}'' - 2A_{23}' \,.\, A_{23}'';$$

or,

$$(x_3x_1 - m'^2x_2^2)(x_1x_2 - m''^2x_3^2) + (x_1x_2 - m'^2x_3^2)(x_3x_1 - m''^2x_2^2)$$
$$- 2(m'^2x_2x_3 - m'x_1^2)(m''^2x_2x_3 - m''x_1^2).$$

Multiplying, and reducing, we get

$$\{2 + 2m'm''(m' + m'')\}\, x_1^2x_2x_3 - (m'^2 + m''^2)x_1(x_2^3 + x_3^3) - 2m'm''x_1^4.$$

Now,

$$2m'm'' = -\frac{1}{m},$$

and

$$m' + m'' = \frac{1}{2m^2}. \qquad \text{(See equations (55), (56).)}$$

Hence we have

$$\frac{1}{4m''}\{(8m'' - 2m)\, x_1^2x_2x_3 - (1 + 4m^3)x_1(x_2^3 + x_3^3) + 4m^3x_1^4\}.$$

Now, since x_1, x_2, x_3, are the co-ordinates of a point on the Hessian, we have

$$x_2^3 + x_3^3 = \frac{1 + 2m^3}{m_2}\, x_1x_2x_3 - x_1^3.$$

Substituting this value, we get

$$\frac{-(1+8m^3)}{4m^6}(x_1^2(x_2x_3 - m^2x_1^2)),$$

or

$$\frac{-(1+8m^3)}{4m^6}\,x_1^2A_{11}\,; \tag{187}$$

hence the squares of the three determinants,

$$\left\| \begin{array}{ccc} A_1', & A_2', & A_3', \\ A_1'', & A_2'', & A_3', \end{array} \right\|$$

are proportional to $x_1^2A_{11}$, $x_2^2A_{22}$, $x_3^2A_{33}$, and the proposition is proved.

217. The first Hessian transformations of the lines LM, NR, are the two pairs of tangents at the two corresponding pairs of points L, M; N, R, respectively: these four tangents meet on the Hessian in a point D (see Fig., Art. 215), which corresponds to the point A, with respect to the first transformation. Similarly, D is the point corresponding to B with respect to the second, and to C with respect to the third transformation. Hence the points D, A, B, C form a quadruple system of points. Again, since the point D is the common tangential of the four points L, M, N, R, the polar conic of D with respect to the Hessian will cut the Hessian in these four points, and the points A, B, C form a self-conjugate triangle with respect to this polar conic. Hence we have the following theorem:—

Any three of the four points of a quadruple system form a self-conjugate triangle of the polar conic of the fourth point with respect to the Hessian.

Cor.—If P and Q be two points on a cubic, and if P be the tangential of Q, then the polar conic of P with respect to the cubic is circumscribed about a self-conjugate triangle with respect to the polar conic of Q. In other words, the polar conic of P is conjugate to the polar conic of Q.

218. The theorem proved in Art. 216 enables us to write down the equations of the six lines joining the four points L, M, N, R, of a

quadruple system; they are as follows, $\alpha_1, \alpha_2, \alpha_3$, being current co-ordinates:

$$
\begin{array}{llllllll}
\text{Equation of} & NR & \text{is} & (x_1A_1)\,\alpha_1 & +\,(x_2A_2)\,\alpha_2 & +\,(x_3A_3)\,\alpha_3 & = & 0. \\
\quad ,, & RL & ,, & (A_1A_1')\,\alpha_1 & +\,(A_2A_2')\,\alpha_2 & +\,(A_3A_3')\,\alpha_3 & = & 0. \\
\quad ,, & MR & ,, & (x_1A_1')\,\alpha_1 & +\,(x_2A_2')\,\alpha_2 & +\,(x_3A_3')\,\alpha_3 & = & 0. \\
\quad ,, & LM & ,, & (A_1'A_1'')\,\alpha_1 & +\,(A_2'A_2'')\,\alpha_2 & +\,(A_3'A_3'')\,\alpha_3 & = & 0. \\
\quad ,, & MN & ,, & (x_1A_1'')\,\alpha_1 & +\,(x_2A_2'')\,\alpha_2 & +\,(x_3A_3'')\,\alpha_3 & = & 0. \\
\quad ,, & NL & ,, & (A_1A_1'')\,\alpha_1 & +\,(A_2A_2'')\,\alpha_2 & +\,(A_3A_3'')\,\alpha_3 & = & 0.
\end{array}
\qquad (188)
$$

This theorem includes that of Art. 212.

219. Since the point A is the intersection of the lines NR and LM: now, denoting LM by λ_α, A is the intersection of the lines

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3,$$

$$(x_1A_1)\,\alpha_1 + (x_2A_2)\,\alpha_2 + (x_3A_3)\,\alpha_3.$$

Hence the co-ordinates of the point A are the determinants

$$\left\| \begin{matrix} \lambda_1, & \lambda_2, & \lambda_3, \\ x_1A_1, & x_2A_2, & x_3A_3, \end{matrix} \right\|$$

but the values of λ_1, λ_2, λ_3, are

$$\left\| \begin{matrix} x_1, & x_2, & x_3, \\ A_1, & A_2, & A_3. \end{matrix} \right\|$$

Hence, after some reductions, and putting $(m - m^4)^{\frac{1}{4}}A_1$, $(m - m^4)^{\frac{1}{4}}A_2$, $(m - m^4)^{\frac{1}{4}}A_3$, for A_1, A_2, A_3—values which will render our formula symmetrical (see also Art. 209, in which it will be seen, if these values were used, the S under the square root would not occur)—we get the co-ordinates of A to be

$$
\begin{aligned}
&(x_2x_3)^2 + (A_{23})^2, \\
&(x_3x_1)^2 + (A_{31})^2, \\
&(x_1x_2)^2 + (A_{12})^2.
\end{aligned}
\qquad (189)
$$

The co-ordinates are expressed in terms of the co-ordinates of the points L and M, and it is plain they may be written as similar functions of the points N and R, thus:

$$\begin{aligned}&(A_{23}')^2 + (A_{23}'')^2,\\ &(A_{31}')^2 + (A_{31}'')^2,\\ &(A_{12}')^2 + (A_{12}'')^2.\end{aligned} \tag{190}$$

The co-ordinates of B may in like manner be expressed in terms of the co-ordinates of L and N, or of those of N and R, and the co-ordinates of C in terms of the co-ordinates of L and K, or of M and N.

220. By reciprocating the result of Art. 212, we have the following theorem, which is also obvious otherwise. *If the polar conic of a point D on the Hessian consists of two lines λ_x, μ_x, then the co-ordinates of D are the products $\lambda_1\mu_1$, $\lambda_2\mu_2$, $\lambda_3\mu_3$.* Now, taking for the lines λ_x, μ_x any two of the three pairs of lines joining the quadruple system $LMNR$ such as LM, NR, and we get the co-ordinates of D to be

$$\begin{aligned}&x_1A_1A_1'A_1'',\\ &x_2A_2A_2'A_2'',\\ &x_3A_3A_3'A_3'',\end{aligned} \tag{191}$$

a system remarkable for its symmetry.

Cor.—Since any cubic may be considered the Hessian of another cubic, we have the following theorem:—*If the co-ordinates of the four points, in which the polar conic of a point P on a cubic cuts the cubic, be x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, then the co-ordinates of P are*

$$\begin{aligned}&xx_1x_2x_3,\\ &yy_1y_2y_3,\\ &zz_1z_2z_3,\end{aligned} \tag{192}$$

respectively.

Cor. 2.—*If the co-ordinates of a point P on the Hessian be the system of determinants,*

$$\left\| \begin{matrix} \lambda_1, & \lambda_2, & \lambda_3, \\ \mu_1, & \mu_2, & \mu_3, \end{matrix} \right\|$$

the co-ordinates of its corresponding point are $\lambda_1\mu_1$, $\lambda_2\mu_2$, $\lambda_3\mu_3$, . . .

221. If we form the polar conic of the point $\lambda_1\mu_1$, $\lambda_2\mu_2$, $\lambda_3\mu_3$, and compare it with the product of the equations of the lines λ_x, μ_x, we get the values of $\lambda_1\mu_1$, $\lambda_2\mu_2$, $\lambda_3\mu_3$ proportional to the expressions

$$\lambda_2\mu_3 + \lambda_3\mu_2, \qquad \lambda_3\mu_1 + \lambda_1\mu_3, \qquad \lambda_1\mu_2 + \lambda_2\mu_1;$$

hence, by *Cor.* 2 of the last Article, we have the following theorem: *The co-ordinates of two corresponding points on the Hessian may be written in the forms*—

$$\begin{aligned} &\lambda_2\mu_3 - \lambda_3\mu_2, \qquad && (\lambda_2\mu_3 + \lambda_3\mu_2); \\ &\lambda_3\mu_1 - \lambda_1\mu_3, \qquad && (\lambda_3\mu_1 + \lambda_1\mu_2); \\ &\lambda_1\mu_2 - \lambda_2\mu_1, \qquad && (\lambda_1\mu_2 + \lambda_2\mu_1). \end{aligned} \tag{193}$$

222. The products $A_1A_1'A_1''$, $A_2A_2'A_2''$, $A_3A_3'A_3''$ admit of simple expression in terms of the co-ordinates x_1, x_2, x_3 of the point x. Thus:—

$$\begin{aligned} A_1^2A_1'^2A_1^2 &= x_2x_3 - m^2x_1^2)(x_2x_3 - m'^2x_1^2)(x_2x_3 - m''^2x_1^2) \\ &= (x_2x_3)^3 - (m^2 + m'^2 + m''^2)x_1^2x_2^2x_3^2 \\ &\quad + (m^2m'^2 + m'^2m''^2 + m''^2m^2)x_1^4x_2^2x_3^2 - m^2m'^2m''^2x_1^6. \end{aligned}$$

Now, from the values given in Articles 80, 81, we get

$$(m^2 + m'^2 + m''^2) = \left(\frac{1 + 2m^3}{2m^2}\right)^2 = \left(\frac{x_1^3 + x_2^3 + x_3^3}{2x_1x_2x_3}\right)^2;$$

by the equation of the Hessian, and

$$m^2m'^2 + m'^2m''^2 + m''^2m^2 = \frac{x_1^3 + x_2^3 + x_3^3}{2x_1x_2x_3};$$

lastly,
$$m^2 m'^2\, m''^2 = \frac{1}{4}.$$

Substituting these values, we get
$$A_1^2 A_1'^2 A_1''^2 = -\left(\frac{x_2^3 - x_3^3}{2}\right)^2:$$

hence the expressions $A_1 A_1' A_1''$; &c., are proportional to—
$$(x_2^3 - x_3^3), \qquad (x_3^3 - x_1^3), \qquad (x_1^3 - x_2^3). \qquad Q.\,E.\,D.$$

Cor.—In like manner the products $x_1 A_1' A_1''$, $x_2 A_2' A_2''$, $x_3 A_3' A_3''$ *are proportional to*
$$(A_2^3 - A_3^3), \qquad (A_3^3 - A_1^3), \qquad (A_1^3 - A_2^3).$$

223. If we substitute the values given in equations (193), Art. 221, in the system of equations (189), we find the co-ordinates of the third point, in which the line joining the two points whose co-ordinates are given in equations (193) meets the Hessian again: thus we get, after an easy calculation,
$$\begin{aligned} x_1 &= (\lambda_2^2 \mu_3^2 - \lambda_3^2 \mu_2^2)^{-1}. \\ x_2 &= (\lambda_3^2 \mu_1^2 - \lambda_1^2 \mu_3^2)^{-1}. \qquad (194) \\ x_3 &= (\lambda_1^2 \mu_2^2 - \lambda_2^2 \mu_1^2)^{-1}. \end{aligned}$$

Cor.—If the co-ordinates of the three points, in which the line joining corresponding points on the Hessian intersects the Hessian be denoted by $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ *respectively, then*
$$\begin{aligned} x_1 y_1 z_1 &= 1. \\ x_2 y_2 z_2 &= 1. \qquad (195) \\ x_3 y_3 z_3 &= 1. \end{aligned}$$

224. If we form the equation of the line joining the points whose co-ordinates are given in the equations (193) we have the following theorem: *If* λ_x, μ_x *be the equations of two corresponding tangents to the*

Cayleyan, the equation of the third tangent, drawn to it from the point of intersection of the lines λ_x, μ_x, *is the determinant*

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ \lambda_2\mu_3 - \lambda_3\mu_2, & \lambda_3\mu_1 - \lambda_1\mu_3, & \lambda_1\mu_2 - \lambda_2\mu_1 \\ \lambda_2\mu_3 + \lambda_3\mu_2, & \lambda_3\mu_1 + \lambda_1\mu_3, & \lambda_1\mu_2 + \lambda_2\mu_1 \end{vmatrix};$$

or the determinant,

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ \lambda_2\mu_3, & \lambda_3\mu_1, & \lambda_1\mu_2 \\ \lambda_3\mu_2, & \lambda_1\mu_3, & \lambda_2\mu_1 \end{vmatrix} = 0; \tag{196}$$

or, expanded,

$$(\lambda_1^2\mu_2\mu_3 - \mu_1^2\lambda_2\lambda_3)x_1 + (\lambda_2^2\lambda_3\mu_1 - \mu_2^2\lambda_3\lambda_1)x_2$$
$$+ (\lambda_3^2\mu_1\mu_2 - \mu_3^2\lambda_1\lambda_2)x_3 = 0. \tag{197}$$

225. If we reciprocate the theorems of Articles 221, 222, we get the following theorem: *The equations of the three tangents which can be drawn from any point x on the Hessian to the Cayleyan may be written in the following form:—*

$$(x_2A_3 - x_3A_2)y_1 + (x_3A_1 - x_1A_3)y_2 + (x_1A_2 - x_2A_1)y_3 = 0;$$
$$(x_2A_3 + x_3A_2)y_1 + (x_3A_1 + x_1A_3)y_2 + (x_1A_2 + x_2A_1)y_3 = 0; \tag{198}$$
$$(x_2^2A_{33} - x_3^2A_{22})^{-1}y_1 + (x_3^2A_{11} - x_1^2A_{33})^{-1}y_2 + (x_1^2A_{22} - x_2^2A_{11})^{-1}y_3 = 0:$$

y_1, y_2, y_3 are current co-ordinates.

226. We have given in Art. 67 the co-ordinates of the nine points of inflection of a cubic, and we are now enabled by Art. 208 to give the co-ordinates of the nine points corresponding to them on the Hessian.

They are as follows: the three points corresponding to points on x_1 are—

$$
\begin{array}{llll}
 & 1).\ 1,\ m,\ m; & 2).\ 1,\ m\omega^2,\ m\omega; & 3).\ 1,\ m\omega,\ m\omega^2; \\
\text{on } x_2, & 4).\ m,\ 1,\ m; & 5).\ m\omega,\ 1,\ m\omega^2; & 6).\ m\omega^2,\ 1,\ m\omega; \\
\text{on } x_3, & 7).\ m,\ m,\ 1; & 8).\ m\omega^2,\ m\omega,\ 1; & 9).\ m\omega,\ m\omega^2,\ 1:
\end{array}
$$

hence, from these values, and the co-ordinates of the points of inflection, we have, by Art. 212, the equations of the nine harmonic polars already given (see Art. 71).

SECTION II.—CORRESPONDING POINTS (*continued*).

227. We have shown in Art. 167 that the bicubic of π—viz.,

$$\frac{d\pi}{da_1}\cdot\frac{d\pi}{dx_1}+\frac{d\pi}{da_2}\cdot\frac{d\pi}{dx_2}+\frac{d\pi}{dx_3}\cdot\frac{d\pi}{dx_4}=0;$$

or, as in Art. 167, say the bicubic $F=0$—possesses the remarkable property of being its own central transformation. We shall now examine this cubic a little more in detail, and for this purpose we shall refer the original or fundamental cubic to a self-conjugate tetragram, which has one side at infinity: thus, let the fundamental cubic be

$$a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3=0,$$

where

$$x_4=-(x_1+x_2+x_3):$$

hence it is clear, since x_4 represents the line at infinity, that our system of co-ordinates must be areal.

228. The Hessian transformation of the line $\lambda_1x_1+\lambda_2x_2+\lambda_3x_3$ is got from equation (90), Art. 112, by making $\lambda_4=0$; then, calling this conic E, we have

$$
\begin{aligned}
E\equiv\ &\lambda_1^2(a_2a_3\,x_2x_3+a_3a_4\,x_3x_4+a_4a_2\,x_4x_2)\\
&+\lambda_2^2(a_3a_4\,x_3x_4+a_4a_1\,x_4x_1+a_1a_3\,x_1x_3)\\
&+\lambda_3^2(a_4a_1\,x_4x_1+a_1a_2\,x_1x_2+a_2a_4\,x_2x_4)\\
&-2a_4x_4(a_1\lambda_2\lambda_3x_1+a_2\lambda_3\lambda_1x_2+a_3\lambda_1\lambda_2x_3)=0: \qquad (199)
\end{aligned}
$$

hence,
$$\frac{dE_1}{dx_1} = \lambda_2^2(a_1a_4x_4 + a_1a_3x_3$$
$$+ \lambda_3^2(a_1a_4x_4 + a_1a_2x_2) - 2a_4a_1\lambda_2\lambda_3x_4;$$

$$\frac{dE_1}{d\lambda_1} = 2\lambda_1(a_2a_3x_2x_3 + a_3a_4x_3x_4 + a_4a_2x_4x_2)$$
$$- 2a_4x_4(a_2\lambda_3x_2 + a_3\lambda_2x_3).$$

Now, since for the line at infinity, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, the equations just written become

$$\frac{d\pi}{dx_1} = a_1(a_2x_2 + a_3x_3)$$

$$\frac{d\pi}{da_1} = 2a_2a_3\,x_2x_3.$$

Hence,
$$\sum_{s=1}^{s=3}\left(\frac{d\pi}{da_s}\cdot\frac{d\pi}{dx_s}\right) = x_2x_3(a_2x_2 + a_3x_3)$$
$$+ x_3x_1(a_3x_3 + a_1x_1) + x_1x_2(a_1x_1 + a_2x_2) = 0. \tag{200}$$

229. The formulæ of central transformation become much simplified when we refer the fundamental cubic to the self-conjugate tetragram having one side at infinity. Thus, let α, β, γ be the co-ordinates of a point, then its polar conic will be

$$\alpha(a_1x_1^2 - a_4x_4^2) + \beta(a_2x_2^2 - a_4x_4^2) + \gamma(a_3x_3^2 - a_4x_4^2);$$

and if the centre of this conic be α', β', γ', the polar line of α', β', γ' will be at infinity; namely, the line $x_1 + x_2 + x_3 = 0$.

Hence,
$$\alpha'\{\alpha(a_1x_1) + a_4(\alpha + \beta + \gamma)x_4\}$$
$$+ \beta'\{\beta(a_2x_2) + a_4(\alpha + \beta + \gamma)x_4\}$$
$$+ \gamma'\{\gamma(a_3x_3) + a_4(\alpha + \beta + \gamma)x_4\}$$
$$= x_1 + x_2 + x_3.$$

Hence, restoring the value of x_4, and comparing coefficients, we get

$$a_1\,\alpha\alpha' - a_4(\alpha+\beta+\gamma)(\alpha'+\beta'+\gamma') = 1\,;$$
$$a_2\,\beta\beta' - a_4(\alpha+\beta+\gamma)(\alpha'+\beta'+\gamma') = 1\,;$$
$$a_3\,\gamma\gamma' - a_4(\alpha+\beta+\gamma)(\alpha'+\beta'+\gamma') = 1.$$

Hence,
$$a_1\,\alpha\alpha' = a_2\beta\beta' = a_3\gamma\gamma'.$$

Let the common value of these products be k, and we have the following formulae of transformation, namely :—

$$\alpha = \frac{k}{a_1\,\alpha'},$$
$$\beta = \frac{k}{a_2\beta'},$$
$$\gamma = \frac{k}{a_3\gamma'}.$$

Hence we change x_1, x_2, x_3 into $\dfrac{k}{a_1x_1}$, $\dfrac{k}{a_2x_2}$, $\dfrac{k}{a_3x_3}$, respectively, and we have the required central transformation. Compare Salmon's *Higher Curves*, page 244.

230. The equation (200), transformed by the formulae of the last Articles, gives

$$\frac{k^2}{a_2a_3\,x_2x_3}\left(\frac{k}{x_2}+\frac{k}{x_3}\right) + \frac{k^2}{a_3a_1\,x_3x_1}\left(\frac{k}{x_3}+\frac{k}{x_1}\right) + \frac{k^2}{a_1a_2\,x_1x_2}\left(\frac{k}{x_1}+\frac{k}{x_2}\right) = 0\,;$$

or,
$$a_1x_1^2\,(x_2+x_3) + a_2x_2^2\,(x_3+x_1) + a_3x_3^2\,(x_1+x_2) = 0\,; \qquad (201)$$

or,
$$x_2x_3(a_2x_2+a_3x_3) + x_3x_1(a_3x_3+a_1x_1) + x_1x_2(a_1x_1+a_2x_2) = 0. \qquad (202)$$

Hence we have an independent proof that the central transformation of the bicubic of π is the bicubic of π.

Cor.—From the form of the equation (201), we see that the tangents

to the bicubic of π, or say the bicubic F at the angular points of the triangle of reference, are

$$x_1 + x_2, \qquad x_2 + x_3, \qquad x_3 + x_1,$$

respectively, and these are parallel to the sides of the triangle of reference, and meet the bicubic again where they meet the line

$$a_1(x_2 + x_3) + a_2(x_3 + x_1) + a_3(x_1 + x_2) = 0. \tag{203}$$

231. The cubic F is the Hessian of the cubic

$$(a_1 + a_2 + a_3)(a_1^2x_1^3 + a_2^2x_2^3 + a_3^2x_3^3) - (a_1x_1 + a_2x_2 + a_3x_3)^3 = 0. \tag{204}$$

This is easily confirmed by calculation, and from it we infer the following theorem:—*The envelope of the lines joining corresponding points on F is a curve of the third class, namely, the Cayleyan of the cubic* (204). In order to give a formal proof of this theorem, we show that if the polar conic of a point $\alpha_1, \alpha_2, \alpha_3$, with respect to the cubic (204), have a double point, that the co-ordinates will be proportional to

$$(a_1\alpha_1)^{-1}, \qquad (a_2\alpha_2)^{-1}, \qquad (a_3\alpha_3)^{-1}.$$

Putting, for shortness, $(a_1 + a_2 + a_3) = d$, the polar conic of $\alpha_1, \alpha_2, \alpha_3$ is

$$\begin{aligned} &\alpha_1\{da_1^2x_1^2 - a_1(a_1x_1 + a_2x_2 + a_3x_3)^2\} \\ +\, &\alpha_2\{da_2^2x_2^2 - a_2(a_1x_1 + a_2x_2 + a_3x_3)^2\} \\ +\, &\alpha_3\{da_3^2x_3^2 - a_3(a_1x_1 + a_2x_2 + a_3x_3)^2)\} = 0. \end{aligned}$$

Now, if this conic has a double point, the co-ordinates of the double point must satisfy its three differentials equated to zero: hence we must have

$$\begin{aligned} \alpha_1 da_1x_1 &= (\alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3)(a_1x_1 + a_2x_2 + a_3x_3), \\ \alpha_2 da_2x_2 &= (\alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3)(a_1x_1 + a_2x_2 + a_3x_3), \\ \alpha_3 da_3x_3 &= (\alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3)(a_1x_1 + a_2x_2 + a_3x_3). \end{aligned}$$

Hence,

$$\alpha_1a_1x_1 = \alpha_2a_2x_2 = \alpha_3a_3x_3;$$

and, therefore, the proposition is proved.

232. *The cubic F is the sum of two cubics, each of which is the central transformation of the other.* Thus the two cubics are

$$a_1 x_1^2 x_2 + a_2 x_2^2 x_3 + a_3 x_3^2 x_1 = 0, \tag{205}$$

$$a_1 x_1^2 x_3 + a_2 x_2^2 x_1 + a_3 x_3^2 x^2 = 0. \tag{206}$$

This proposition is evident, and we see why F is its own transformation.

Cor.—The difference of the cubics (205), (206), namely,

$$a_1^2 x_1^2 (x_2 - x_3) + a_2 x_2^2 (x_3 - x_1) + a_3 x_3^2 (x_1 - x_2) = 0; \tag{207}$$

is another cubic which is its own transformation.

233. If we combine the equation of F with the equation of the line

$$x_1 = kx_2,$$

we find that the pair of lines

$$(a_1 k^2 + a_2 k) x_2^2 + (a_1 k^2 + a_2) x_2 x_3 + a_3 (1 + k) x_3^2 = 0,$$

passes through their intersection. Hence, if $x_1 - kx_2 = 0$, be a tangent to F, this pair of lines must be a perfect square; and, therefore, we must have

$$(a_1 k^2 + a_2)^2 - 4 (a_1 k^2 + a_2 k)(a_3 + a_3 k) = 0;$$

or,

$$(a_1 k^2 - 2a_3 k + a_2)^2 - 4 \{ a_3 (a_1 + a_2 + a_3) \} k^2 = 0.$$

Hence, eliminating k between this and the equation $x_1 = kx_2$, we get the equation of the four tangents from the principal point (12) to F, viz., these are

$$(a_1 x_1^2 - 2a_3 x_1 x_2 + a_2 x_2^2)^2 - 4 \{ a_1 + a_2 + a_3) a_3 \} x_1^2 x_2^2 = 0. \tag{208}$$

234. The equation (208), being the difference of two squares, breaks up into two factors,

$$a_1 x_1^2 - 2 (a_3 \pm \sqrt{a_3 (a_1 + a_2 + a_3)}) x_1 x_2 + a_2 x_2^2 = 0. \tag{209}$$

Each of these factors represents a pair of tangents; and, transforming by

central transformation, each factor remains unaltered, or each is transformed into itself. Hence, the points of contact of each pair with F are corresponding points. Hence we have the following theorem:—*The points of contact of the four tangents, which can be drawn to the bi-cubic of π from any of the principal points, consist of two pairs of corresponding points.*

235. We find, in a similar way, the equation of the four tangents from the principal point (12) to the curve (207) to be

$$(a_1x_1^2 - 2a_3x_1x_2 + a_2x_2^2)^2 + 4(a_2 - a_3)(a_3 - a_1)x_1^2x_2^2 = 0; \qquad (210)$$

and we see in this case also that two of the points of contact are the correspondents of the other two.

Section III.—Equations of Poles.

236. We have, in Art. 142, equations (120), (121), given the equation of the four poles of the lines μ_x. We shall, in this section, solve a more special problem, namely, we shall find the equation of the four poles of a line which is a tangent to the Cayleyan; or, in other words, which joins two corresponding points on the Hessian. Let (a_1, a_2, a_3), (A_1, A_2, A_3), be the co-ordinates of the two corresponding points. Now, the polar conic of the first, with respect to the fundamental cubic, is

$$(a_1, a_2, a_3, ma_1, ma_2, ma_3)(x_1, x_2, x_3)^2 = 0;$$

and therefore the equation of the four poles will be the envelope of the system of conics—

$$(a_1 + kA_1, a_2 + kA_2, a_3 + kA_3, m(a_1 + kA_1), m(a_2 + kA_2), m(a_3 + kA_3))(x_1x_2x_3)^2 = 0,$$

where k has any value: hence, the required envelope is

$$\Phi^2 = 4\Sigma\Sigma' = 0, \qquad (211)$$

where Σ, Σ' are the results of substituting the co-ordinates a_1, a_2, a_3;

A_1, A_2, A_3 respectively in the Hessian transformation of λ_x, and Φ is the contravariant:—

$$(a_2A_3 + a_3A_2 - 2m^2a_1A_1)\lambda_1^2 + 2m\{m(a_2A_3 + a_3A_2) - 2a_1A_1\}\lambda_2\lambda_3$$
$$+ (a_3A_1 + a_1A_3 - 2m^2a_2A_2)\lambda_2^2 + 2m\{m(a_3A_1 + a_1A_3) - 2a_2A_2\}\lambda_3\lambda_1$$
$$+ (a_1A_2 + a_2A_1 - 2m^2a_3A_3)\lambda_3^2 + 2m\{m(a_1A_2 + a_2A_1) - 2a_3A_3\}\lambda_1\lambda_2 = 0. \quad (212)$$

237. The last equation can be written in a very simple form. In order to show this, I observe that $m(a_2A_3 + a_3A_2)$ is $= -a_1A_1$; for, multiplying by A_1, we get

$$m(a_2A_{31} + a_3A_{12}) = -a_1A_{11};$$

and, substituting the values of A_{31}, A_{12}, A_{11}, we see that this equality holds in virtue of the equation

$$-m^2(a_1^3 + a_2^3 + a_3^3) + (1 + 2m^3)a_1a_2a_3 = 0:$$

hence, the equation of Φ may be written

$$\Phi \equiv (1 + 2m^3)\{a_1A_1\lambda_1^2 + a_2A_2\lambda_2^2 + a_3A_3\lambda_3^2\}$$
$$+ 6m^2\{a_1A_1\lambda_2\lambda_3 + a_2A_2\lambda_3\lambda_1 + a_3A_3\lambda_3\lambda_1\} = 0. \quad (213)$$

238. Since the equation of the line joining the two remaining points of the quadruple system to which the points (a_1, a_2, a_3), (A_1, A_2, A_3) belong, is

$$a_1A_1x_1 + a_2A_2x_2 + a_3A_3x_3 = 0$$

(see Art. 212); then, if this line be denoted by u_x, the equation of Φ may be written

$$\Phi \equiv (1 + 2m^3)\{u_1\lambda_1^2 + u_2\lambda_2^2 + u_3\lambda_3^2\}$$
$$+ 6m^2\{u_1\lambda_2\lambda_3 + u_2\lambda_3\lambda_1 + u_3\lambda_1\lambda_2\} = 0. \quad (214)$$

239. Since Σ, see equation (211), is the result of substituting a_1, a_2, a_3, which are the co-ordinates of a point on the Hessian, in the equation of the Hessian transformation of λ_x, it is by the theory of determinants equal to the negative square of $(\lambda_1A_1 + \lambda_2A_2 + \lambda_3A_3)$, or $= -(\lambda_A)^2$ (see Salmon's

Modern Algebra, page 32, Art. 37). In like manner, Σ' is equal $-S\lambda_a^2$, where $S = m - m^4$: hence, the tangential equation of the four poles of the line joining the points (a_1, a_2, a_3), (A_1, A_2, A_3) on the Hessian, is the product of the factors

$$\Phi \pm 2S^{\frac{1}{2}}\lambda_a \,.\, \lambda_A = 0. \tag{215}$$

240. Let A and B be the corresponding points on the Hessian (see Fig., Art. 159); then the equations of these points are λ_a, λ_A respectively, and their polar conics are the pairs of lines

$$BL,\ BN; \qquad AR,\ AN;$$

then Φ is the envelope of lines which cuts these four lines harmonically. Hence it is a conic inscribed in the quadrilateral $LMRN$, and touches its four sides in the points a, b, c, d, which are the fourth harmonics on each side of the points on that side which are its intersection with the three remaining sides. Hence, the triangle formed by the points A, B, and their common tangential point D, is self-conjugate with respect to Φ.

241. The discriminant of Φ is the determinant

$$\begin{vmatrix} ka_1A_1, & a_3A_3, & a_2A_2 \\ a_3A_3, & ka_2A_2, & a_1A_1 \\ a_2A_2, & a_1A_1, & ka_3A_3 \end{vmatrix} = 0, \tag{216}$$

where we put for shortness $k = \dfrac{1 + 2m^3}{3m^2}$. Hence, expanding, we have the discriminant of Φ given by the equation

$$(2 + k^3)a_1a_2a_3A_1A_2A_3 = k\{(a_1A_1)^3 + (a_2A_2)^3 + (a_3A_3)^3\};$$

and, squaring, we get

$$\begin{aligned}&(2 + k^3)^2 a_1^2a_2^2a_3^2A_{11} \,.\, A_{22} \,.\, A_{33} \\ &= k^2\{a_1^6A_{11}^3 + a_2^6A_{22}^3 + a_3^6A_{33}^3 + 2a_2^3a_3^3A_{23}^3 + 2a_3^3a_1^3A_{31}^3 + 2a_1^3a_2^3A_{12}^3\}.\end{aligned} \tag{217}$$

We shall, in the last Chapter, give the geometrical interpretation of the expressions on both sides of this equation.

242. The discriminant of Φ may be also written (see Art. 238) in the form

$$\begin{vmatrix} ku_1, & u_3, & u_2 \\ u_3, & ku_2, & u_1 \\ u_2, & u_1, & ku_3 \end{vmatrix} = 0. \qquad (218)$$

This equation denotes a curve of the third class, inscribed in the figure formed by the nine cuspidal tangents of the Cayleyan.

SECTION IV.—TANGENTIALS.

243. We see, from Articles 220, 222, that if the co-ordinates of any point on the Hessian be x_1, x_2, x_3, that the co-ordinates of its tangential are

$$x_1(x_2^3 - x_3^3),$$
$$x_2(x_3^3 - x_1^3),$$
$$x_3(x_1^3 - x_2^3).$$

If in this system we replace x_1, x_2, x_3 by A_1, A_2, A_3, or A_1', A_2'', A_3'', or, finally, by A_1'', A_2'', A_3'' (see Art. 222), the values remain unaltered; showing, as we know otherwise, that the four points of a quadruple system have the same tangential.

244. We can find, by means of the values given in the last Article, the co-ordinates of the n^{th} tangential of the point x as follows; putting, for shortness,

$$x_2^3 - x_3^3 = X_1, \qquad x_3^3 - x_1^3 = X_2, \qquad x_3^3 - x_2^3 = X_3,$$

and denoting by accents the co-ordinates of the tangential, we have

$$x_1' = x_1X_1,$$
$$x_2' = x_2X_2,$$
$$x_3' = x_3X_3.$$

Hence the co-ordinates of the second tangential are

$$x_1'' = x_1X_1(x_2^3X_2^3 - x_3^3X_3^3) = x_1X_1X_1' \text{ suppose,}$$
$$x_2'' = x_2X_2(x_3^3X_3^3 - x_1^3X_1^3) = x_2X_2X_2' \quad ,,$$
$$x_3'' = x_3X_3(x_1^3X_1^3 - x_2^3X_2^3) = x_3X_3X_3' \quad ,,$$

In like manner the co-ordinates of the third tangential are

$$x_1''' = x_1X_1X_1'(x_2^3X_2^3X_2'^3 - x_3^3X_3^3X_3'^3) = x_1X_1X_1'X_1'' \text{ suppose,}$$
$$x_2''' = x_2X_2X_2'(x_3^3X_3^3X_3'^3 - x_1^3X_1^3X_1'^3) = x_2X_2X_2'X_2'' \quad ,,$$
$$x_3''' = x_3X_3X_3'(x_1^3X_1^3X_1'^3 - x_2^3X_2^3X_2'^3) = x_3X_3X_3'X_3'' \quad ,,$$

Hence, in general,

$$\begin{aligned} x_1^{(n)} &= x_1X_1X_1'X_1'' \;\ldots\; X_1^{(n-1)}, \\ x_2^{(n)} &= x_2X_2X_2'X_2'' \;\ldots\; X_2^{(n-1)}, \\ x_3^{(n)} &= x_3X_3X_3'X_3'' \;\ldots\; X_3^{(n-1)}, \end{aligned} \tag{219}$$

where the law of formation is evident.

245. From the foregoing system of values of the tangentials we can find the conditions that a point shall coincide with its own n^{th} tangential. Thus, if a point coincide with its own third tangential, we have, ρ being any constant—

$$X_1X_1'X_1'' - \rho = 0,$$
$$X_2X_2'X_2'' - \rho = 0,$$
$$X_3X_3'X_3'' - \rho = 0:$$

hence,

$$\rho\left(\frac{1}{X_1X_1'} + \frac{1}{X_2X_2'} + \frac{1}{X_3X_3'}\right) = X_1'' + X_2'' + X_3'';$$

but

$$X_1'' + X_2'' + X_3'' = 0,$$

by the law of formation;

$$\therefore \frac{1}{X_1X_1'} + \frac{1}{X_2X_2'} + \frac{1}{X_3X_3'} = 0. \tag{220}$$

In like manner, if a point coincide with its own fourth tangential, we have

$$\frac{1}{X_1X_1'X_1''} + \frac{1}{X_2X_2'X_2''} + \frac{1}{X_3X_3'X_3''} = 0\,; \tag{221}$$

and, in general, if a point coincide with its n^{th} tangential,

$$\sum_{s=1}^{s=3}\left(\frac{1}{X_sX_s'X_s'' \;.\;.\;.\;.\; X_s^{(n-2)}}\right) = 0. \tag{222}$$

246. If the equation (220) be cleared of fractions, and the values restored of X_1, X_1', &c., we find the result equal to the product of the factor

$$x_1^6 + x_2^6 + x_3^6 - x_2^3x_3^3 - x_3^3x_1^3 - x_1^3x_2^3\,; \tag{223}$$

and the expression

$$\begin{aligned}(x_1^3 - x_2^3)^2 (x_2^3 - x_3^3)^2 (x_3^3 - x_1^3)^2 (x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) \\ + (x_1^3 - x_2^3)^3 (x_1^3 - x_3^3)^3 (x_2^3x_3^3) + (x_2^3 - x_3^3)^3 (x_2^3 - x_1^3)^3 (x_3^3x_1^3) \\ + (x_3^3 - x_1^3)^3 (x_3^3 - x_2^3)^3 (x_1^3x_2^3).\end{aligned} \tag{224}$$

247. The expressions (224) may be written also in the form

$$\left(\frac{x_1\,x_2}{X_1}\right)^3 + \left(\frac{x_3\,x_3}{X_1}\right)^3 + \left(\frac{x_3\,x_1}{X_2}\right)^3 = \frac{(x_1'x_2)^3 + (x_2\,x_3)^3 + (x_3x_1)^3}{X_1X_2X_3}\,;$$

or still in the form

$$\begin{aligned}(x_1^3 - x_2^3)^2 + (x_2^3 - x_3^3)^2 + (x_3^3 - x_1^3)^2 \\ \times \{x_1^6x_2^3 + x_2^6x_3^3 + x_3^6x_1^3 - 3x_1^3\,x_2^3\,x_3^3\} \\ \times \{x_1^3\,x_2^6 + x_2^3\,x_3^6 + x_3^3\,x_1^6 - 3x_1^3\,x_2^3\,x_3^3\} = 0.\end{aligned} \tag{225}$$

This latter is Dr. Salmon's form, who arrived at it by a different method (see *Proceedings of the Royal Society*, vol. ix., page 333). The extraneous factor (223) is the same as the first of the three factors (225), so that the intersections of this with the Hessian do not give any points distinct from

those given by the equation (225). I have not succeeded in accounting, geometrically, why this factor should occur twice.

The equation (225) being of the 24th degree, it will intersect the Hessian in seventy-two points, and these coincide with their third tangentials; and, since it is independent of m, it represents the locus of all such points for all the curves of the syzygetic pencil of cubics passing through the nine points of inflection.

The invariant equation

$$\Theta^4 + 54S\Theta^2H^4 - 27T\Theta H^6 - 243S^2H^8 = 0;$$

which Dr. Hart works with in his Paper on the "Nine Point Contact of Cubic Curves" (see *Transactions of the Royal Irish Academy*, vol. xxv., page 559), coincides with the above only for points on the fundamental cubic.

248. The equation (224) may be written in the form

$$\begin{aligned}&(x_1X_1)^3(x_2X_2)^3 + (x_2X_2)^3(x_3X_3)^3 + (x_3X_3)^3(x_1X_1)^3 \\ &\quad + (x_1^3 - x_2^3)^2(x_2^3 - x_3^3)^2(x_3^3 - x_1^3)^2(x_1^3x_2^3 + x_3^3x_1^3 + x_1^3x_2^3). \qquad (226)\end{aligned}$$

Now in forming the Hessian transformation of (226), we remark that the Hessian transformation of $x_1X_1 = m\,x_1X_1$, and of $x_2X_2 = m\,x_2X_2$, &c., we find the Hessian transformation to be the curve of 24th degree—

$$\begin{aligned}&m^6\{(x_1X_1)^3(x_2X_2)^3 + (x_2X_2)^3(x_3X_3)^3 + (x_3X_3)^3(x_1X_1)^3\} \\ &+ (A_{11}^3 - 2A_{12}^3 + A_{22}^3)(A_{22}^3 - 2A_{23}^3 + A_{33}^3)(A_{33}^3 - 2A_{31}^3 + A_{11}^3)(A_{12}^3 + A_{23}^3 + A_{31}^3) \\ &= 0. \qquad (227)\end{aligned}$$

Since the curve (225) *meets the Hessian in seventy-two points, which are nine-pointic contacts for cubic curves, the curve* (227) *meets the Hessian in the seventy-two corresponding points, and these are points of eighteen-pointic contact for the Hessian with sextic curves.*

249. The second Hessian transformation of the curve (226) is

$$m'^6 \{(x_1X_1)^3(x_2X_2)^3 + (x_2X_2)^3(x_3X_3)^3 + (x_3X_3)^3(x_1X_1^3)^3\}$$
$$+ (A_{11}'^3 - 2A_{12}'^3 + A_{22}'^3)(A_{22}'^3 - 2A_{23}'^3 + A_{33}'^3)(A_{33}'^3 - 2A_{31}'^3 + A_{11}'^3)(A_{12}'^3 + A_{23}'^3 + A_{31}'^3)$$
$$= 0; \qquad (228)$$

and the third is

$$m''^6 \{(x_1X_1)^3(x_2X_2)^3 + (x_2X_2)^3(x_3X_3)^3 + (x_3X_3)^3(x_1X_1)^3\}$$
$$+ (A_{11}''^3 - 2A_{12}''^3 + A_{22}''^3)(A_{22}''^3 - 2A_{23}''^3 + A_{33}''^3)(A_{33}''^3 - 2A_{31}''^3 + A_{11}''^3)(A_{12}''^3 + A_{23}''^3 + A_{31}''^3)$$
$$= 0. \qquad (229)$$

Hence, there are on the Hessian two hundred and sixteen points, each of which is a point of eighteen-pointic contact with sextic curves; and since any cubic may be considered as the Hessian of some curve, we see that the property is true for every curve of the third order.

CHAPTER VIII.

250. We shall devote this Chapter to some problems and miscellaneous matter that could not be conveniently put in otherwise.

251. If the Hessian transformation of a line be an equilateral hyperbola, find the envelope of the line. Writing the Hessian transformation of λ_x in the form

$$(a, b, c, f, g, h)(x_1, x_2, x_3)^2 = 0,$$

the condition that it represents an equilateral hyperbola is

$$a + b + c - 2f \cos A - 2g \cos B - 2h \cos C = 0,$$

where A, B, C denote the angles of the triangle of reference; and, restoring

the values of a, b, c, &c., we get a result which may be written

$$m^2(1,\ 1,\ 1,\ \cos A,\ \cos B,\ \cos C)(\lambda_1,\ \lambda_2,\ \lambda_3)^2$$
$$+ (\cos A,\ \cos B,\ \cos C,\ m,\ m,\ m)(\lambda_1,\ \lambda_2,\ \lambda_3)^2 = 0. \qquad (230$$

Hence, four equilateral hyperbolas can be described through a given point, to have triple contact with the Hessian. For, substituting the co-ordinates of the given point in the Hessian transformation of λ_x, the result will be the tangential equation of a conic, which conic will have with the conic (230) four common tangents.

252. If the Hessian transformation of λ_x be a parabola, its envelope will be the curve of fourth class—

$$(\epsilon_{11},\ \epsilon_{22},\ \epsilon_{33},\ \epsilon_{23},\ \epsilon_{31},\ \epsilon_{12})(\sin A,\ \sin B,\ \sin C)^2 = 0, \qquad (231)$$

where ϵ_{11}, ϵ_{22}, &c., have the values given in Art. 59, equation (60). Hence, reasoning as in the last Article, *we infer that eight parabolas can be described through a given point to have triple contact with a cubic.*

253. If the equation (230), which represents the condition for a hyperbola, be represented by Ξ, and the equation (231) by Ξ', the envelope of λ_x, when its Hessian transformation is a circle, will be the curve of fourth class,

$$\Xi^2 = 4\Xi' \qquad (232)$$

(see Salmon's *Conics*, 6th edition, page 352). *Hence, eight circles can be described through a given point to have triple contact with the Hessian.*

Cor.—From the results of the three last Articles, combined with equation (120), *we infer that eight equilateral hyperbolas, sixteen circles, and sixteen parabolas can be described, having triple contact with the Hessian, and touching a given line; or, in other words, Chasles' Characteristics for conics having triple contact with the Hessian are*—

Equilateral hyperbola,	$\mu = 4,$	$\nu = 8,$
Parabola,	$\mu = 8,$	$\nu = 16,$
Circle,	$\mu = 8,$	$\nu = 16.$

254. We have seen that the Hessian transformation of any curve of odd degree touches the Hessian of the fundamental cubic in points which correspond to the points of intersection of the original curve with the Hessian. Hence, as a special case, we have the following theorem: *The Hessian transformation of any curve of the syzygetic pencil*

$$x_1^3 + x_2^3 + x_3^3 + kx_1x_2x_3,$$

when k has any value, will touch the Hessian in the nine points corresponding to the nine points of inflection. Again, since every quadratic function of the curves $x_1^3 + x_2^3 + x_3^3 = 0$, $x_1x_2x_3 = 0$, passes twice through each point of inflection, we see that the Hessian transformation of such function touches the Hessian in the nine points corresponding to the points of inflection. *Hence the Hessian transformation of each of the covariants* (E_s, E_t, ///) *where s and t may take all values from* 1 *to* 9 *inclusive* (see Art. 187), *is a sextic touching the Hessian in nine points.*

255. The curve $x_1x_2x_3$ is one of the curves of the pencil

$$x_1^3 + x_2^3 + x_3^3 + kx_1x_2x_3 = 0,$$

namely, the one which corresponds to $k = \infty$. Its Hessian transformation is

$$A_{11} \,.\, A_{22} \,.\, A_{33};$$

or

$$(x_2x_3 - m^2x_1^2)(x_3x_1 - m^2x_2^2)(x_1x_2 - m^2x_3^2).$$

Now H denoting the Hessian, and putting, for shortness, $V = x_1^2x_2^2x_3^2$, $W = x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3$, we find, after some easy reduction,

$$A_{11} \,.\, A_{22} \,.\, A_{33} = m^4W - m^3(2 + m^3)V + Hx_1x_2x_3. \tag{233}$$

256. The curve of the pencil which corresponds to $k = 0$ is $x_1^3 + x_2^3 + x_3^3$, and its transformation is

$$(A_1^3 + A_2^3 + A_3^3)^2,$$

or

$$A_{11}^3 + A_{22}^3 + A_{33}^3 + 2A_{12}^3 + 2A_{23}^3 + 2A_{31}^3 = 0.$$

Hence, restoring values, we get the required transformation—

$$\frac{(1+2m^3)^2}{m}\{mW-(2+m^3)V\}$$
$$+\frac{2(1-m^3)^2}{m}Hx_1x_2x_3-\frac{(2+m^3)H_2}{m}. \qquad (234)$$

257. Lastly, let us take the simplest quadratic function of $x_1^3+x_2^3+x_3^3$ and $x_1x_2x_3$, namely, their product, and we find the Hessian transformation

$$A_{11}\,.\,A_{12}\,.\,A_{13}+A_{22}\,.\,A_{21}\,.\,A_{23}+A_{33}\,.\,A_{31}\,.\,A_{32}$$
$$=m(1+2m^3)\{mW-(2+m^3)V\}+m(2+m^3)Hx_1x_2x_3. \qquad (235)$$

Cor.—We may infer from each of the three last Articles that the curve

$$mW-(2+m^3)V=0 \qquad (236)$$

touches the Hessian in the nine points which correspond to the nine points of inflection.

258. The Hessian transformation of W, or

$$x_1^3x_2^3+x_2^3x_3^3+x_3^3x_1^3$$

is

$$A_{12}^3+A_{23}^3+A_{31}^3=0;$$

and, restoring values, we get

$$m^3(2+m^3)W-\frac{(1+m^3+7m^6)}{m}V$$
$$+\frac{1}{m}\{(2+m^3)Hx_1x_2x_3-H^2\}=0. \qquad (237)$$

Cor.—By means of the results of Articles 255, 258, we get the Hessian transformation of

$$mW-(2+m^3)V$$

to be

$$=-\{H^2+(1-m^3)^3V\};$$

and this denotes, as it ought, a curve touching the Hessian in the nine points of inflection.

259. If we denote the covariant $mW - (2 + m^3)V$ by Π, we easily find its expression in terms of Dr. Salmon's fundamental covariants, U, H, Θ, thus:

$$(1 + 8m^3)^2 m^3 \Pi = 2(m^3 - 1)H^2 - m^2(5 + 4m^3)UH - m\Theta; \tag{238}$$

or

$$m\Theta = 2(m^3 - 1)H^2 - m^2(5 + 4m^3)UH - (1 + 8m^3)^2 m^3 \Pi. \tag{239}$$

Hence we have the following important theorem: *The covariant Θ touches the Hessian in the nine points which correspond to the nine points of inflection.*

Cor.—Θ is the Hessian transformation of a sextic touching the Hessian at the nine points of inflection. We shall give this transformation in Art. 263.

260. Since the covariant Π touches the Hessian at the nine points corresponding to the nine points of inflection, then, by the method explained in the Chapter on reciprocation, we have the following theorem:— *The contravariant*

$$4m^2(\lambda_1^3\lambda_2^3 + \lambda_2^3\lambda_3^3 + \lambda_3^3\lambda_1^3) - (1 - 16m^3)\lambda_1^2\lambda_2^2\lambda_3^2 \tag{240}$$

touches the Cayleyan at the points of contact of the nine tangents which correspond to the cuspidal tangents.

261. In the same manner we get from the covariant Θ the following contravariant

$$\begin{aligned}(1 - 16m^3)\Psi^2 + 2(1 - 4m^3)\Psi K + 3K^2 \\ - 64S^2(\lambda_2^3\lambda_3^3 + \lambda_3^3\lambda_1^3 + \lambda_1^3\lambda_2^3) = 0.\end{aligned} \tag{241}$$

This is the condition that the conics E_4, E_6 (see Chapter VI.), considered as conics in point co-ordinates, should be cut harmonically by the line λ_x. This contravariant also touches the Cayleyan at the points of contact of the nine tangents corresponding to the cuspidal tangents.

262. The function $\frac{dU}{dx_1}\cdot\frac{dH}{dx_1}+\frac{dU}{dx_2}\cdot\frac{dH}{dx_2}+\frac{dU}{dx_3}\cdot\frac{dH}{dx_3}$ remains unaltered by Hessian transformation. *Hence the twelve points in which this quartic cuts the Hessian are such that six of them are points corresponding to the remaining six;* and, by reciprocation, *we see that the curve of fourth class,*

$$\frac{d\Psi}{d\lambda_1}\cdot\frac{dK}{d\lambda_1}+\frac{d\Psi}{d\lambda_2}\cdot\frac{dK}{d\lambda_2}+\frac{d\Psi}{d\lambda_3}\cdot\frac{dK}{d\lambda_3} = 0, \qquad (242)$$

is touched by six pairs of corresponding tangents to the Cayleyan; or, in other words, this curve is its own Cayleyan transformation.

263. From Articles 43, 44, we infer that the Hessian transformation of

$$SU^2 + 12H^2$$

is

$$= -4S\Theta.$$

Hence we have the following theorem: *The covariant Θ is all to a numerical factor the Hessian transformation of*

$$SU^2 + 12H^2. \qquad (243)$$

264. We have seen, Art. 243, that if the co-ordinates of any point on the Hessian be x_1, x_2, x_3, the co-ordinates of its first tangential are $x_1(x_2^3 - x_3^3)$, $x_2(x_3^3 - x_1^3)$, $x_3(x_1^3 - x_2^3)$. Hence, if in the equation of any curve ϕ we replace x_1, x_2, x_3 by $x_1(x_2^3 - x_3^3)$, &c., we get the equation of a new curve which intersects the Hessian in points which are the tangentials of the points in which ϕ intersects it. Now, let us denote the result of this transformation on W by W' and on V by V' (see Art. 255), and then we have

$$W' \quad \text{or} \quad x_1'^3x_2'^3 + x_2'^3x_3'^3 + x_3'^3x_1'^3$$

$$= -\{x_1^3x_2^3(x_1^3 - x_3^3)^3(x_2^3 - x_3^3)^3 + x_2^3x_3^3(x_2^3 - x_1^3)^3(x_2^3 - x_3^3)^3$$
$$+ x_3^3x_1^3(x_3^3 - x_2^3)^3(x_1^3 - x_2^3)^3\}.$$

In like manner

$$V' \text{ or } (x_1' x_2' x_3')^2$$
$$= x_1^2 x_2^2 x_3^2 (x_1^3 - x_2^3)^2 (x_2^3 - x_3^3)^2 (x_3^3 - x_1^3)^2.$$

From these values we see that the determinant

$$\begin{vmatrix} W, & V, \\ W', & V', \end{vmatrix}$$

or
$$(x_1^3 x_2^3 + x_2^3 x_3^3 + x_3^3 x_1^3)(x_1' x_2' x_3')^2$$
$$- (x_1'^3 x_2'^3 + x_2'^3 x_3'^3 + x_3'^3 x_1'^3)(x_1 x_2 x_3)^2, \qquad (244)$$

is equal to the product of the equation (224), which determines the points on a cubic which coincide with their third tangentials, by the factor $(x_1 x_2 x_3)^2$. This gives a very compact form of that equation.

265. The equation (220) may be written in the form

$$\frac{x_1}{x_1 X_1 X_1'} + \frac{x_2}{x_2 X_2 X_2'} + \frac{x_3}{x_3 X_3 X_3'};$$

or, denoting the co-ordinates of the second tangentials of the point x_1, x_2, x_3 by x_1'', x_2'', x_3'', as in Art. 245, and we have the following simple form

$$\frac{x_1}{x_1''} + \frac{x_2}{x_2''} + \frac{x_3}{x^{3''}} = 0. \qquad (245)$$

It would scarcely be possible to imagine a simpler form than this.

Or, since the point x_1, x_2, x_3 is its own third tangential, it must be the first tangential of the point x_1'', x_2'', x_3''. Hence, from the relation connecting a point with its first tangential, we have another method of getting the equation (245).

266. The equation (220) gives also the following elegant result (see Fig., Art. 215). Let L be the point x_1, x_2, x_3, and let the co-ordinates of the points M, N, R, which with L form a quadruple system, be α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3, respectively. Again, let the co-ordinates of the points

A, B, C, which are the points of intersection of the three line pairs through L, M, N, R, or, in other words, the points which with the tangential of L form a quadruple system, be α_1', β_1', γ_1'; α_2', β_2', γ_2'; α_3', β_3', γ_3'; then it is easy to see that we have the equation

$$X_1 = \alpha_1 \alpha_2 \alpha_3, \qquad X_2 = \beta_1 \beta_2 \beta_3, \qquad X_3 = \gamma_1 \gamma_2 \gamma_3,$$

$$X_1' = \alpha_1' \alpha_2' \alpha_3', \qquad X_2' = \beta_1' \beta_2' \beta_3', \qquad X_3' = \gamma_1' \gamma_2' \gamma_3'.$$

Then the equation (220) may be written in the following symmetrical manner:—

$$(\alpha_1 \alpha_2 \alpha_3 \alpha_1' \alpha_2' \alpha_3')^{-1} + (\beta_1 \beta_2 \beta_3 \beta_1' \beta_2' \beta_3')^{-1} + (\gamma_1 \gamma_2 \gamma_3 \gamma_1' \gamma_2' \gamma_3')^{-1} = 0. \quad (246)$$

267. All the equations that we have given for third tangentials and the Hessian transformation of these equations have corresponding contravariants, viz., these are obtained by writing λ_1, λ_2, λ_3 for x_1, x_2, x_3, &c.; thus, the Cayleyan transformation of the contravariant which corresponds to equation (220) is

$$\begin{aligned} &(\kappa_{11}^3 + \kappa_{23}^3 - \kappa_{12}^3 - \kappa_{31}^3)^3 \kappa_{23}^3 \\ &+ (\kappa_{22}^3 + \kappa_{31}^3 - \kappa_{23}^3 - \kappa_{12}^3)^3 \kappa_{31}^3 \\ &+ (\kappa_{33}^3 + \kappa_{12}^3 - \kappa_{31}^3 - \kappa_{23}^3)^3 \kappa_{12}^3 \end{aligned}$$

$$+(\kappa_{11}^3+\kappa_{22}^3-2\kappa_{12}^3)(\kappa_{22}^3+\kappa_{33}^3-2\kappa_{23}^3)(\kappa_{33}^3+\kappa_{11}^3-2\kappa_{31}^3)(\kappa_{12}^3+\kappa_{23}^3+\kappa_{31}^3)=0, \quad (247)$$

when κ_{11}, κ_{22}, &c., denote the minors of the Cayleyan.

268. If a_1, a_2, a_3 be the co-ordinates of a point which corresponds to a point of inflection on the Hessian, the tangent at this point, viz.,

$$x_1 \left(\frac{dH}{dx_1}\right) + x_2 \left(\frac{dH}{dx_2}\right) + x_3 \left(\frac{dH}{dx_3}\right) = 0,$$

must pass through the corresponding point. Hence we must have

$$(H_1 A_1 + H_2 A_2 + H_3 A_3) = 0,$$

where H_1, H_2, H_3 denote differentials, and A_1, A_2, A_3 have the usual umbral meaning. Hence, squaring, we get

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & H_1 \\ u_{21}, & u_{22}, & u_{23}, & H_2 \\ u_{31}, & u_{32}, & u_{33}, & H_3 \\ H_1, & H_2, & H_3 & \end{vmatrix} = 0. \qquad (248)$$

This covariant expresses the condition that the Hessian transformation of the polar line of any point may pass through the point; and from the mode in which we arrived at it, we see that it touches the Hessian in the nine points which correspond to the nine points of inflection.

269. By reciprocating from the last Article, we have the following contravariant:—

$$\begin{vmatrix} -2m\lambda_1, & \lambda_3, & \lambda_2, & \dfrac{dK}{d\lambda_1} \\ \lambda_3, & -2m\lambda_2, & \lambda_1, & \dfrac{dK}{d\lambda_2} \\ \lambda_2, & \lambda_1, & -2m\lambda_3, & \dfrac{dK}{d\lambda_3} \\ \dfrac{dK}{d\lambda_1}, & \dfrac{dK}{d\lambda_2}, & \dfrac{dK}{d\lambda_3}, & 0 \end{vmatrix} = 0. \qquad (249)$$

This contravariant touches the Cayleyan at the nine points of contact of the nine tangents which correspond to the cuspidal tangents.

270. We have, in Art. 241, equation (216), given the discriminant of the conic Φ, which is the envelope of lines cutting two polar conics harmonically, each polar conic consisting of a line pair. Now, if this discriminant vanishes, it is easy to see that one of the conics must pass through the double point of the other. Hence, if the discriminant vanishes, the two conics must be the polar conic of a point of inflection and the polar conic

of its corresponding point. Hence, the co-ordinates of either point must reduce the equation (217) to an identity: hence, substituting x_1, x_2, x_3 for a_1, a_2, a_3, &c., we have the following theorem: The curve of twelfth degree,

$$\begin{aligned} &x_1^6(x_2x_3 - m^2x_1^2)^3 + 2(x_2^3x_3^3)(m^2x_2x_3 - mx_1^2)^3 \\ &+ x_2^6(x_3x_1 - m^2x_2^2)^3 + 2(x_3^3x_1^3)(m^2x_3x_1 - mx_2^2)^3 \\ &+ x_3^6(x_1x_2 - m^2x_3^2)^3 + 2(x_1^3x_2^3)(m^2x_1x_2 - mx_3^2)^3 = 0, \end{aligned} \tag{250}$$

meets the Hessian in all the points where the curve

$$x_1^2x_2^2x_3^2(x_2x_3 - m^2x_1^2)(x_3x_1 - m^2x_2^2)(x_1x_2 - m^2x_3^2) = 0$$

meets it; but this latter meets the Hessian twice in each of the nine points of inflection, and twice in each of the corresponding points. Hence the former curve (250) touches the Hessian in these eighteen points.

271. The equation (250) may be written very simply as follows, viz.: put a, b, c, f, g, h to denote the cubes of the minors of the Hessian; then the equation (250) is

$$(\mathrm{a, b, c, f, g, h})(x_1^3, x_2^3, x_3^3)^2 = 0, \tag{251}$$

and this curve touches the Hessian in the nine points of inflection and in their nine corresponding points.

272. If the equation (250) be multiplied out, we find, after re-arranging,

$$\begin{aligned} &(1 - 2m^3)(x_1x_2x_3)^3(x_1^3 + x_2^3 + x_3^3) - 3m^2(x_1x_2x_3)^2(x_1^3 + x_2^3 + x_3^3)^2 \\ &+ m^6(x_1^3 + x_2^3 + x_3^3)(x_1^3 + x_2^3 - x_3^3)(x_1^3 - x_2^3 + x_3^3)(x_2^3 + x_3^3 - x_1^3) \\ &+ 6m^2(1 - m^3)(x_1x_2x_3)^2(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) \\ &+ 3m^4(x_1x_2x_3)(x_1^9 + x_2^9 + x_3^9) + 18m^4(x_1x_2x_3)^4 = 0. \end{aligned} \tag{252}$$

Now we have

$$\begin{aligned} x_1^9 + x_2^9 + x_3^9 = &(x_1^3 + x_2^3 + x_3^3)^3 + 3(x_1x_2x_3)^3 \\ &- 3(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3)(x_1^3 + x_2^3 + x_3^3); \end{aligned}$$

and it is easy to verify that the coefficient of

$$m^6 \text{ is } = -(x_1^3 + x_2^3 + x_3^3)^4 - 8(x_1^3 + x_2^3 + x_3^3)(x_1x_2x_3)^3$$
$$+ 4(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3)(x_1^3 + x_2^3 + x_3^3)^2.$$

Now, putting for shortness $Z = x_1^3 + x_2^3 + x_3^3$, and if W and V have the same signification as in recent Articles, the equation (252) may be written

$$(1 - 2m^3 - 8m^6)V^{\frac{2}{3}}Z - 3m^2VZ^2$$
$$+ m^6(4W - Z^2)Z^2 + 6m^2(1 - m^3)WV$$
$$+ 3m^4V^{\frac{1}{3}}\{Z^3 + 9V^{\frac{2}{3}} - 3WZ\} = 0;$$

and this equation can be expressed in terms of the fundamental covariants U, H, Θ, as follows: for

$$V^{\frac{1}{3}} = \frac{m^2U + H}{1 + 8m^3}, \qquad Z = \frac{(1 + 2m^3)U - 6mH}{(1 + 8m^3)},$$

$$W = \frac{m^3(2 + m^3) - m(1 + 2m^3)UH + 3m^2H^2 - \Theta}{(1 + 8m^3)^2}.$$

Substituting these values, rejecting the denominator $(1 + 8m^3)^4$, which will be common, and changing sign, we get

$$m^7(1 - 57m^3 + 90m^6 - 64m^9)U^3H$$
$$+ 3m^5(1 - 13m^3 - 50m^6 + 160m^9)U^2H^2$$
$$- (1 + 12m^3 + 99m^6 + 680m^9 + 1152m^{12})UH^3$$
$$+ 3m(2 - 17m^3 + 152m^6 + 288m^9)H^4$$
$$+ m^6(1 - 12m^3 + 16m^6)U^2\Theta - 3m^4(15 - 8m^3 + 32m^6)UH\Theta$$
$$+ 6m^2(1 + 8m^3 + 24m^6)H^2\Theta = 0. \qquad (253)$$

In this covariant, if we put $H = 0$, it reduces to $U^2\Theta$, and since U meets H in the nine points of inflection, and Θ touches it in the corresponding points, *we see that our covariant touches the Hessian in eighteen points.*

273. In the equation (253), if we put $U = 0$, we see that our covariant touches U in the nine points of inflection; and, since it touches H in the same points, *it follows that these points are double points on the covariant.* I shall denote this covariant by the letter J.

274. The covariant J has a corresponding contravariant, thus: denoting the cubes of the minors of the Cayleyan by the Greek $\alpha, \beta, \gamma, \delta, \epsilon, \phi$, the contravariant will be

$$(\alpha, \beta, \gamma, \delta, \epsilon, \phi)(\lambda_1{}^3\lambda_2{}^3\lambda_3{}^3)^2 = 0\,; \tag{254}$$

and, if we substitute values, this becomes

$$\begin{aligned} &16m^3(1 + 4m^3)(\lambda_1\lambda_2\lambda_3)^3(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3) \\ &- 48m^2(\lambda_1\lambda_2\lambda_2)^2(\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)^2 \\ &+ (\lambda_1{}^3 + \lambda_2{}^3 + \lambda_3{}^3)(\lambda_1{}^3 + \lambda_2{}^3 - \lambda_3{}^3)(\lambda_1{}^3 - \lambda_2{}^3 + \lambda_3{}^3)(\lambda_2{}^3 + \lambda_3{}^3 - \lambda_1{}^3) \\ &+ 12m(1 + 8m^3)(\lambda_1\lambda_2\lambda_3)^2(\lambda_1{}^3\lambda_2{}^3 + \lambda_2{}^3\lambda_3{}^3 + \lambda_1{}^3\lambda_2{}^3) \\ &+ 12m(\lambda_1\lambda_2\lambda_3)(\lambda_1{}^9 + \lambda_2{}^9 + \lambda_3{}^9 + 6\lambda_1{}^3\lambda_2{}^3\lambda_3{}^3) = 0. \end{aligned} \tag{255}$$

This contravariant touches the Cayleyan at the nine cusps, and at the points of contact of the nine tangents which correspond to the cuspidal tangents. The cuspidal tangents of the Cayleyan, moreover, are double tangents to the curve, the other points being the points of contact of these tangents with the contravariant Ψ (see Art. 63).

CORRIGENDA.

Page 1, line 1, *for* "A is conjugate to B," *read* "A is self-conjugate with respect to B."
,, 5, ,, 15, ,, "$\lambda' x$, $\lambda'' x''$," *read* "λ'_x, λ''_x."
,, 6, lines 9 and 10, *instead of* two thick vertical lines, *there should be* two pair of thin vertical lines.
,, 8, line 14, *for* "$-\lambda_1\alpha_1 + (\lambda_2\alpha_2 + \lambda_3\alpha_3)L$," *read* "$-(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3)L$."
,, 9, ,, 14, ,, "at," *read* "of."
,, 9, ,, 17, ,, "the point (α)," *read* "the point of intersection of the lines L and M."
,, 15, ,, 19, ,, "conjugate," *read* "consecutive."
,, 17, ,, 10, *Insert the words* "of the Hessian."
,, 27, ,, 7, *Insert the word* "according," at the beginning of this line.
,, 53, ,, 1, *Insert the word* "be."
,, 64, *For the last sentence in this page read* "the equation (109) is a necessary but not a sufficient condition for a quartic to break up into a line and a cubic."
,, 65, line 17, *for* "in," *read* "on."
,, 81, ,, 14, *omit the word* "properly."
,, 112, ,, 20, *for* "formula" *read* "formulae."
,, 131, ,, 7, ,, "when," *read* "where."

JULY, 1886.]

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ETC., ETC.

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CHIMPANZEE, ORANG-UTAN, AND GIBBON.

BY

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"CUNNINGHAM MEMOIRS."

The Lumbar Curve in Man and the Apes, with an Account of the Topographical Anatomy of the Chimpanzee and Orang-utan. By D. J. CUNNINGHAM, M.D. (Edin. & Dubl.); M.R.I.A.; Professor of Anatomy and Chirurgery in the University of Dublin. (With PLATES I. to XI.).

[Read, February 8th, 1886.]

THE structural differences between Man and the anthropoid Apes are very largely due to man's assumption of the erect attitude, and to his having dispensed with the use of his upper limbs as a means of locomotion. Thus the upper limbs are shortened, whilst the lower limbs are lengthened and strengthened, and possess the power of being fully extended at the hip and knee-joints. There is not a region of the body in which modifications in structure to suit this characteristic posture may not be found; and it is natural that the vertebral column, in its vertical position, should exhibit peculiarities which are not shared by those animals in which it is oblique or more or less horizontal. A feature which has attracted a considerable amount of attention is the lumbar curve—that bulging or convexity forwards in the region of the loins which constitutes so marked a character of the human spine. By some this curve is regarded as peculiar to man, and this view has recently received the support of Pansch[1] and Aeby.[2] Other anatomists deny to man the prescriptive

[1] "Anatomische Vorlesungen," von Dr. Ad. Pansch, 1884.
[2] "Beiträge zur Osteologie des Gorilla." Morph. Jahr. vol. iv.

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right to such a distinction, and consider that he shares it with certain of the anthropoid apes. The observations which I have made in this direction tend to minimize the importance of the lumbar curve as a distinctive character of any special group. Not only do the higher apes possess this curve, but so also do the majority of the lower apes; even some of the quadrupeds, under certain conditions, may show traces of it. In man and the Chimpanzee the quality of the curve is identical; the only difference is in the extent of the curvature and in its period of development. But further investigations which I have made upon several of the low races (although necessarily of a very imperfect character from the want of fresh spines) would seem to indicate that the curve is not stamped upon the vertebral column of, say, the Andaman, Australian, or Negro, so firmly as it is stamped upon the column of the European. I do not mean to infer that the degree of curvature is less in these races; but whereas in the European the bodies of the vertebræ are more or less moulded in adaptation to the curve, in the low races there is not a trace of this.

In prosecuting this inquiry into the conditions of the lumbar curve I have approached the question from two different points of view. In the first place I have examined the form-adaptations which are presented by the bodies of the vertebræ concerned; and in the second place I have endeavoured to determine the quality and the degree of the curve in fresh spines. The first part of this Memoir, therefore, is divided into two sections, corresponding to the results obtained in each of these fields.

Section I.

ADAPTATION OF THE VERTEBRAL BODIES WITH REFERENCE TO THE LUMBAR CURVE.

This section includes the results obtained by an extended series of measurements of the bodies of the lower five true vertebræ in Europeans, five of the lower races of man, the anthropoid apes, and some of the lower apes of the Old World. European vertebræ have been measured by various anatomists, but always on a very limited scale. Thus in the works of the

brothers Weber,[1] Nühn,[2] Hirschfeld,[3] Horner,[4] Meyer,[5] Barwell,[6] and Aeby,[7] the adaptation of the vertebral bodies to the different curves is treated with more or less fulness: further, Aeby has compared the measurements of a European spine with those of the vertebral bodies of a gorilla, two gibbons, and some of the lower apes. None of these investigators, however, have extended their observations in this direction to the lower races of man and all the anthropoid apes.

Method adopted in making the Measurements.—The only form-adaptations of the vertebral bodies which could contribute to the formation of the antero-posterior lumbar curve are those which affect the anterior and posterior vertical diameters of the centra, or, in other words, those which produce the wedge-shape or "keilform" condition. Consequently the measurements were strictly confined to the determination of the vertical depth of the body of each of the five lower true vertebræ in front and behind, and to a comparison of the results obtained in each case. The instruments which were employed were Professor Flower's craniometer and the French *compas glissière.* But in several instances it was necessary to measure the bones in the articulated skeleton. This was avoided wherever it was possible to do so, because the determination of the posterior vertical diameters of the vertebral bodies in such cases is a matter of great difficulty; and even with the utmost care errors are apt to occur. A few of the lower races and of the anthropoid apes were measured under these conditions, and to ensure accuracy a special pair of calipers were used. In

[1] "Mechanik der menschlichen Gehwerkzeuge," von den Brüdern Wilhelm und Eduard Weber (1836), p. 92.

[2] "Untersuchungen und Beobachtungen aus dem Gebiete der Anatomie," &c. Erstes Heft (1849), p. 14.

[3] "Nouvel aperçu sur les conditions anatomiques des courbures de la colonne vertébrale chez l'homme." Gaz. Méd. de Paris. 1849.

[4] "Ueber die Normale Krümmung der Wirbelsäule." Müller's Archiv für Anat. Phys., &c. (1854), p. 478.

[5] "Lehrbuch der physiologischen Anatomie des Menschen." 1861.

[6] "The Causes and Treatment of Lateral Curvature of the Spine." 1868.

[7] "Beiträge zur Osteologie des Gorilla." Morph. Jahr., vol. iv.

these the limbs were of such a size that they could readily be introduced into the spinal canal through the intervertebral foramina, and the extremities of the limbs were bent to the side at a right angle, in such a fashion that they could readily grasp the upper and lower borders of the vertebral body. It is indeed unfortunate, that in the articulation of skeletons of anthropological interest the custom of introducing leather, or cement, or other material to represent the cartilaginous discs, should be so prevalent.

Index of the Vertebral Bodies.—In comparing the results obtained by these measurements, and in striking an average for each race and group examined, it is convenient to construct an index for the separate vertebræ, and also for the sum of the vertebral measurements in a given individual. In calculating such an index I have adopted the anterior vertical diameter of the vertebral body as the standard. The index for an individual vertebra can be determined as follows:—

$$\frac{\text{Posterior vertical diameter} \times 100.}{\text{Anterior vertical diameter.}}$$

Then the combined index, expressive of the condition of the group of vertebræ in a particular individual, may be obtained thus:—

$$\frac{\text{Sum of posterior measurements} \times 100.}{\text{Sum of the anterior measurements.}}$$

The result arrived at by this latter calculation may be termed the *lumbo-vertebral index.*

From the above it must be clear that a vertebra with an index of 100 may be considered to be neutral; it is equally deep in front and behind, and can in no way contribute to the formation of a curve. A vertebra, on the other hand, with an index of 100 +, is moulded in a manner unfavourable to the formation of a curve with the convexity looking forwards; it is deeper behind than in front: whilst a vertebra with an index of 100 – has its body adapted to the lumbar curve; it is deeper in front than behind. The same holds good for the average of the indices of the five lower true vertebræ, which is expressed by the term of lumbo-vertebral index.

Results derived from the Measurements.—The following Table will show the general results of the measurements. In the gorilla, chimpanzee, and orang, which have four lumbar vertebræ, and the lower apes, which have six or seven lumbar vertebræ, the five lower true vertebræ have been considered comparable with the five lumbar vertebræ of man:—

TABLE A.

		MAN.						APES.							
		76 Europeans.	17 Australians.	3 Tasmanians.	3 Bushmen.	23 Andamans.	10 Negroes.	5 Gorillas.	9 Chimpanzees.	4 Orangs.	6 Gibbons.	2 Baboons.	3 Macaques.	1 Colobus.	1 Semnopithecus.
Five Lower true Vertebræ.	*a*	106·1	119·8	115·1	115·9	112·6	113·5	115·3	125·3	113·7	112·8	117·7	109·7	103·8	108·1
	b	101·4	113·0	109·9	113·4	111·2	111·3	111·7	117·1	118·9	108·8	120·6	107·8	103·8	112·7
	c	97·2	113·6	110·1	109·9	108·1	105·9	111·3	116·4	119·7	107·5	108·0	103·0	103·8	112·2
	d	93·5	103·9	109·5	100·8	102·6	105·1	105·3	116·1	111·9	106·4	107·3	103·2	108·3	102·4
	e	81·6	90·4	92·4	95·3	91·4	92·0	101·9	115·8	103·5	104·1	92·8	96·2	90·4	89·2
Lumbo-vertebral Index, . . .		95·8	107·8	107·2	106·6	104·8	105·4	108·1	117·5	112·9	107·1	108·5	103·7	102·4	105·1

The difference between the European lumbo-vertebral index and the corresponding indices of the lower races is very remarkable. In the former it is 95·8, indicating, therefore, a condition of the vertebral bodies favourable to the formation of a curve in a forward direction. In the lower races the lowest lumbo-vertebral index is that of the Andamans, viz. 104·8. In these races, therefore, the lumbar vertebræ, as a group, show no adaptation in form to the characteristic curve in this region, but are moulded upon an opposite plan.

A comparison of the indices of the individual vertebræ is no less striking. In the European the first lumbar vertebra is the only member of the group with an index (106·1), which shows a decided preponderance of the posterior vertical diameter over the anterior vertical diameter; the second

vertebra exhibits a tendency in the same direction, but to all intents and purposes it is neutral. The lower three vertebræ have all indices well below the standard, and are therefore adapted to the lumbar curve.

Very different is the condition of the separate vertebræ in the low races. In these the only lumbar vertebra, with a centrum possessing an index below the standard, and thus fashioned in a manner favourable to the lumbar curve, is the last member of the series. The upper four show a reverse condition. Whilst this is the case, however, it will be observed that, with the exception of the Tasmanians and Australians, the indices of the lumbar vertebræ diminish regularly from above downwards.

But the Table brings out the relation which the anthropoid apes bear to man in this respect. Between these and the Europeans there is a wide gap. On the other hand, it will be noticed, that in so far as the general lumbo-vertebral index is concerned, there is virtually no distinction between the gorilla and gibbon with their indices of 108·1 and 107·1, and the Australians and Tasmanians, with indices of 107·8 and 107·2. In the chimpanzee and orang the lumbo-vertebral index is very considerably higher. It is interesting to note this difference between the chimpanzee and gorilla—it is a point of some significance in the determination of the curve in the latter.

In another particular the Australians and the Tasmanians show a resemblance to the anthropoid apes. In these the indices of the second and third lumbar vertebræ are very similar, whilst, as we have noted, the indices of the other races diminish regularly from above downwards.

The index of the last lumbar vertebra in the anthropoid ape is widely divergent from that in all the human races, and this must be regarded as a matter of primary importance. In all the anthropoid apes it is well above the standard; in man, the wedge-shaped appearance of the body of the last lumbar vertebra is a matter of everyday knowledge. It is interesting to note that this essential character is not nearly so strongly marked in the low races as in the European.

As this point became developed, in the course of my work, I believed that I had arrived at a feature which might be considered to be a distinctly human characteristic—one which was not shared by any other member of

the order Primates. An examination of the lumbar region of certain of the lower apes proved that in this I was mistaken. In the Baboon, the Macaque, Semnopithecus, and in Colobus the last lumbar vertebra possesses an index (as will be seen in the Table) in every respect comparable with that of the lower races, although it is not so low as that of the European. Aeby[1] had previously noted this point in Papio sphinx, and in Macacus nemestrinus, and he ascribes it to the right cause. It has little or nothing to do with the lumbar curve, but is produced by the marked dorsal tilting of the pelvis in these animals. The marked discrepancy between the anterior and posterior vertical diameters of this vertebra is, therefore, chiefly due to a sloping of the inferior surface. In Plates iv. and vi., in which are represented reduced tracings of mesial sections of the frozen Macacus rhesus and Cynocephalus anubis, this condition will be observed. The oblique position of the pelvis is very evident.

The corresponding condition of the last lumbar vertebra in man is due to the same cause, viz. the great backward sweep of the sacrum; and the fact that in the low races this character is not so prominent might almost lead us to infer that the pelvic inclination is not so marked as in the European. This is a point, however, which could only be determined by the examination of fresh spines, because deficiencies in the body of the last lumbar vertebra may be compensated for by an increased anterior depth of the lumbo-sacral cartilaginous disc.

The relationships which have been stated above can be more easily compared, and more fully appreciated, by referring to Plate i., Chart A, in which the conditions of the vertebral bodies are expressed diagrammatically. The column of figures on each side are the indices; the five main vertical subdivisions are allotted, from left to right, to the five lower true vertebræ, whilst in the sixth is given the average lumbo-vertebral index. Each of these compartments is further subdivided into columns corresponding to the number of animals under examination. A strong horizontal line at the level of the standard index (100) divides the chart into an upper and lower portion, and therefore a single glance is sufficient for the recognition of the

[1] "Beiträge zur Osteologie des Gorilla." Morph. Jahr. vol. iv.

manner in which the vertebral bodies are moulded with reference to the lumbar curve. To avoid complexity, only four races of man, and three anthropoid apes, are introduced into this chart. The dotted lines represent the apes; the solid lines refer to man.

The striking manner in which the European stands apart in this respect from his lower brethren, and also from the anthropoid apes, is rendered very evident by this chart. Only two of the European vertebræ are above the standard line, whilst the fourth vertebra occupies a place in the lower portion of the chart almost as low as the last lumbar vertebra of the Australian and Andaman. On the other hand, it will be observed, that the orang and chimpanzee lines stand aloof from the Australian and Andaman, but in an opposite direction. The gorilla line intersects, and is closely associated with, those of the low races. The sudden descent of the latter below the standard constitutes the distinguishing character—the fourth lumbar vertebra of the Australian and Andaman occupying a level very similar to that of the last lumbar vertebra of the gorilla.

SPECIAL POINTS IN CONNEXION WITH EACH RACE.

Europeans.—The foregoing averages were obtained from the measurements of seventy-six European spines, viz. forty-three Irish, twenty-six French, three English, and four German. The detailed indices, resulting from each measurement, are given in the subjoined Tables.

Table B.

INDICES OF LUMBAR VERTEBRÆ IN FORTY-THREE IRISH SPINES.

	TWENTY-ONE MALES.																					21 Males. Average Index.
I. Lumbar Vertebra, .	103·5	104·3	111·0	113·2	106·0	112·0	106·0	105·6	103·7	111·1	109·2	109·0	98·0	96·4	100·0	103·8	120·0	105·3	115·3	100·0	112·0	106·9
II. Lumbar Vertebra, .	100·0	102·0	93·3	100·0	103·7	103·7	103·8	101·7	105·3	105·1	107·1	93·5	100·0	100·0	94·9	103·8	118·5	100·0	111·1	101·9	94·5	102·0
III. Lumbar Vertebra, .	96·5	96·0	88·5	89·8	96·1	105·8	94·4	98·3	100·0	90·6	101·6	98·3	100·0	91·6	96·6	108·0	103·5	103·4	100·0	94·7	98·0	97·7
IV. Lumbar Vertebra, .	96·5	84·6	87·0	87·7	102·0	100·0	86·2	96·6	93·3	87·5	95·0	90·4	86·6	93·2	86·2	103·8	107·1	101·7	93·1	100·0	96·5	95·0
V. Lumbar Vertebra, .	72·0	74·0	82·4	79·3	83·0	85·7	68·9	80·0	90·0	84·3	83·6	85·7	89·6	91·3	84·6	75·8	90·9	89·2	72·5	89·6	79·0	82·4
Lumbo-vertebral Index, . .	93·4	91·7	92·0	93·6	97·8	101·1	91·1	96·1	98·2	95·0	98·9	95·0	94·6	94·4	92·5	98·4	107·7	100·0	97·5	97·0	95·6	96·2

	TWENTY-TWO FEMALES.																						22 Females. Average Index.
I. Lumbar Vertebra,	104·0	103·7	105·5	105·7	105·8	97·9	103·5	104·0	104·0	100·0	93·2	104·0	96·6	101·9	110·2	102·1	106·3	101·8	101·8	104·0	100·0	102·0	102·6
II. Lumbar Vertebra,	100·0	103·4	101·8	98·2	100·0	100·0	96·7	96·2	100·0	100·0	89·2	98·1	91·9	101·8	103·9	96·0	96·0	100·0	96·5	100·0	94·6	94·3	98·1
III. Lumbar Vertebra,	89·3	100·0	100·0	94·8	100·0	92·0	103·3	96·2	98·1	91·2	103·4	96·4	90·4	96·4	92·8	86·7	94·3	94·4	89·8	98·0	91·2	92·8	95·0
IV. Lumbar Vertebra,	84·2	84·7	96·0	91·5	96·3	88·4	96·6	98·0	94·2	90·9	83·3	96·4	93·1	98·1	92·5	86·7	86·7	88·4	91·2	103·8	88·2	91·6	91·8
V. Lumbar Vertebra,	77·4	82·7	86·0	83·3	92·3	76·9	70·0	75·4	82·6	80·3	74·1	83·9	65·5	83·0	85·7	78·8	81·4	85·1	81·9	85·7	81·0	83·7	81·2
Lumbo-vertebral Index, . . .	90·5	94·7	97·7	94·4	98·8	90·8	93·9	93·8	95·7	92·3	88·5	95·5	87·7	96·2	96·6	89·8	92·6	93·9	92·0	98·0	90·5	93·0	93·5

Table C.

INDICES OF LUMBAR VERTEBRÆ IN TWENTY-SIX FRENCH SPINES.—(Both Sexes).

																											Average Index.
I. Lumbar Vertebra,	112·5	108·0	111·0	107·0	113·0	111·5	107·4	113·0	100·0	103·8	108·0	105·7	103·8	108·0	107·4	105·4	111·5	102·0	103·7	112·7	111·1	112·0	103·7	123·8	108·0	112·0	**108·6**
II. Lumbar Vertebra,	100·0	96·3	111·5	100·0	108·0	104·0	100·0	108·0	93·0	100·0	98·0	101·8	112·2	110·0	101·7	103·5	109·0	100·0	98·2	104·0	103·4	103·6	101·7	109·6	98·1	103·7	**103·0**
III. Lumbar Vertebra,	92·0	96·5	107·7	93·5	96·0	103·5	100·0	96·5	89·6	96·5	96·5	100·0	98·2	92·5	100·0	100·0	109·0	96·2	95·0	100·0	105·2	101·9	93·2	103·7	96·4	107·5	**98·7**
IV. Lumbar Vertebra,	89·0	96·5	93·0	84·0	82·7	100·0	100·0	96·5	89·6	96·5	80·7	94·0	101·8	89·0	98·0	96·5	103·7	100·0	95·0	98·1	100·0	100·0	88·3	101·8	96·1	103·7	**95·1**
V. Lumbar Vertebra,	72·4	89·3	83·0	76·7	65·5	80·6	79·3	83·3	85·0	89·3	79·3	85·7	77·5	83·6	85·0	84·7	82·1	88·4	82·5	80·7	93·1	75·7	84·7	79·6	77·1	94·5	**82·2**
Lumbo-vertebral Index,	92·5	97·0	99·2	92·0	91·5	99·2	97·0	98·5	91·4	97·0	93·8	98·0	99·5	96·2	98·5	97·8	103·0	97·3	94·5	98·8	102·4	98·1	94·0	103·0	94·5	104·1	**97·2**

Table D.

INDICES OF LUMBAR VERTEBRÆ IN THREE ENGLISH SPINES.

	2 Males.		Average.	1 Female.[2]	Average. 2 Males, 1 Female.
I. Lumbar Vertebra, ...	118·0	101·0[1]	109·5	101·8	**106·9**
II. Lumbar Vertebra, ...	101·7	107·0	104·3	111·0	**106·5**
III. Lumbar Vertebra, ...	101·7	101·7	101·7	103·7	**102·3**
IV. Lumbar Vertebra, ...	98·2	91·5	94·8	87·2	**92·3**
V. Lumbar Vertebra, ...	79·0	81·5	80·2	76·0	**78·8**
Lumbo-vertebral Index, ...	99·0	96·5	97·7	96·0	**97·1**

[1] The male spine in the Cambridge Museum, which was embedded in plaster of Paris, and then cut by Professor Humphry, in the mesial plane.
[2] The female spine in the same Museum, which had been prepared in a similar manner.

Table E.

INDICES OF LUMBAR VERTEBRÆ IN FOUR GERMAN SPINES.

	3 Males.				1 Female.	4 Spines.
	Braune.	Aeby.[1]	Brothers Weber.	Result.	Braune.	Both Sexes.
I. Lumbar Vertebra, ...	103·9	103·6	107·6	105·0	100·0	103·8
II. Lumbar Vertebra, ...	96·5	103·6	107·4	102·5	101·9	102·3
III. Lumbar Vertebra, ...	88·3	95·0	98·2	93·8	92·3	93·4
IV. Lumbar Vertebra, ...	93·4	93·2	91·2	92·6	80·3	89·5
V. Lumbar Vertebra, ...	67·7	83·3	80·0	77·0	81·4	78·1
Lumbo-vertebral Index, ...	89·3	95·4	96·4	93·7	91·0	93·0

[1] It is hardly right to include this spine with those of the Germans, as in all probability it was taken from a Swiss subject.

The measurements of the Irish spines were made upon fresh spines obtained in the Practical Anatomy Rooms of Trinity College, Dublin. In each case a mesial section was made by means of a saw. The vertical depth of the bodies of the vertebræ in front and behind could then be measured with the greatest accuracy.

The French vertebræ were measured in the dry macerated condition, and chiefly from spines which had been prepared by M. Tramond of Paris. Care was taken to eliminate all those which bore signs of disease, or which were composed of vertebræ which were suspected to have been derived from different individuals.

The measurements of the English spines were obtained in the Anatomical Museum of the Cambridge Museum, and they include the two spines from which Professor Humphry determined the vertebral curves after the manner of the brothers Weber. One of these spines is figured in his admirable work on the Human Skeleton.

The sources from which I derived the results relating to the German spines are indicated in Table E. In the two large Plates by Braune, representing mesial sections of a male and female, the vertebral bodies were carefully measured. As these drawings were built up from tracings taken from the frozen subject, there is every reason to believe that they are thoroughly reliable, even in such small details as those which affect the anterior and posterior vertical depth of the vertebral bodies. In another case the lumbar vertebræ, in the large plate which accompanies the classical work of the brothers Weber,[1] were measured, whilst the indices of the fourth were calculated from measurements given by the late Professor Aeby, in his Paper upon the "Osteology of the Gorilla."[2]

By the following analysis the individual peculiarities of the lumbar vertebræ in this large number of specimens may be the more easily appreciated :—

TABLE F.

SEVENTY-SIX EUROPEAN SPINES.

	Anterior depth of Centrum greater than posterior depth.	Anterior depth equal to posterior depth.	Posterior depth of Centrum greater than anterior depth.
I. Lumbar Vertebra, .	5	6	65
II. Lumbar Vertebra, .	21	17	38
III. Lumbar Vertebra, .	48	11	17
IV. Lumbar Vertebra, .	57	7	12
V. Lumbar Vertebra, .	76	—	—

But it may be asked, Do any of the European spines show indices which can be compared with the indices of the low races? In only six instances have I met with a European spine with a lumbo-vertebral index above the standard, viz. two Irish and four French—the highest being that of an Irish male, with a lumbo-vertebral index of 107·7. It is further interesting to note that, all through, the French indices are higher than the Irish.

[1] *Antea*, p. 3. [2] *Antea*, p. 3.

At this stage it would be well to take note of the observations which have been made in this field by other investigators. In every case the measurements have been confined to Europeans, and the number of individuals examined has been very small. The brothers Weber,[1] who may be said to be the first anatomists to give accurate information upon the normal curves of the human spine, measured the centra of one vertebral column, and from the indices of the lumbar vertebræ of this spine, which are given in Table E, it will be seen that the results which they obtained are very similar to those given in the foregoing Tables. According to Horner[2] there is no regularity in the vertebral column in this respect, except in the case of the lower lumbar vertebræ. "Whilst everywhere," he says, "there is seen a slightly different height, sometimes in front, and at other times behind, there happens here a constant stronger height in front—a smaller behind;" and he goes on to remark, "that on the average all the loin vertebræ contribute to the curve. Of a surety, therefore, this curve is the most permanent, and the most pronounced." This statement he supports by giving the measurements of the five lumbar centra in five individuals, and his results, in so far as the excess of the anterior depth of the vertebral bodies over the posterior depth is concerned, are somewhat in advance of mine. The average index calculated from his figures is :—

I. Lumbar vertebra, ...	98·5
II. Lumbar vertebra, ...	97·4
III. Lumbar vertebra, ...	93·8
IV. Lumbar vertebra, ...	88·9
V. Lumbar vertebra, ...	81·3
Lumbo-vertebral Index, . .	91·9

The index of the first lumbar vertebra in these measurements is peculiar, in so far as it is below the standard. In only five out of the seventy-six spines I examined did I find this to be the case

Nühn[3] arrived at somewhat different results. He divided the vertebral

[1] *Antea*, p. 3. [2] *Antea*, p. 3. [3] *Antea*, p. 3.

columns of several well-built male subjects in their whole length, and then measured upon the surface of the section the fore and hinder surfaces of all the vertebral bodies, and he concludes, "that the bodies of the lumbar vertebræ can have no share in the production of the lumbar curve," as the upper four are deeper behind than in front. He only gives the detailed measurements of one specimen, and from his figures I have calculated the following indices :—

I. Lumbar vertebra, ...	108·4
II. Lumbar vertebra, ...	100·0
III. Lumbar vertebra, ...	99·0
IV. Lumbar vertebra, ...	104·3
V. Lumbar vertebra, ...	71·0
Lumbo-vertebral Index, . . .	96·5

Whilst, therefore, the general lumbo-vertebral index is very similar to that indicated in Table B., p. 9, yet the index of the fourth lumbar vertebra is exceptionally high. In only one case out of the seventy-six spines that I examined is the index of this vertebra above 104.

Hirschfeld,[1] also, by measurements of the vertebral bodies, denies that in any region these contribute to the production of the curves, and in this he is supported by H. Meyer,[2] who found no difference whatever in the healthy vertebral column between the fore and hind surfaces of the vertebral bodies, with the exception of the last lumbar and the first sacral vertebræ; and the form of these, he asserts, is determined by the inclination of the sacrum. Barwell gives expression to similar views.

In three different publications the late Professor Aeby of Bern refers to this point, viz. in his text-book, entitled, "Der Bau des menschlichen Körpers;" in his Paper upon the "Osteology of the Gorilla," which has already been quoted;[3] and in his elaborate memoir upon "Die Altersvershiedenheiten der menschlichen Wirbelsaüle."[4] In the last article he

[1] *Antea*, p. 3. [2] *Antea*, p. 3. [3] *Antea*, p. 3.

[4] "Archiv für Anatomie und Entwicklungsgeschichte," His und Braune (1879).

gives the results he obtained from the measurements of eight spines, classified in a Table constructed on a similar plan to Table F, p. 12. I take the liberty of subjoining this Table for the purposes of comparison :—

EIGHT SPINES MEASURED BY PROFESSOR AEBY.

	Anterior depth greater than posterior depth.	Anterior depth equal to posterior depth.	Anterior depth less than posterior depth.
I. Lumbar Vertebra, .	1	3	4
II. Lumbar Vertebra, .	1	6	1
III. Lumbar Vertebra, .	4	3	1
IV. Lumbar Vertebra, .	7	1	—
V. Lumbar Vertebra, .	8	—	—

His results, therefore, are virtually the same as mine.

Lower Races of Man.—Five of the lower races of man are represented in Table A, p. 5, which I have given to show the general results. In all, fifty-six separate spines were examined, viz. of twenty-three Andamans, seventeen Australians, ten Negroes, three Tasmanians, and three Bushmen. And, in recording the separate indices of these, I must give expression to my sense of the deep obligations under which I lie to Dr. J. G. Garson, Sub-Curator of the Royal College of Surgeons Museum, London; to Professor Macalister of Cambridge; to Mr. More, Curator of the Natural History Department of the Science and Art Museum in Dublin; and to Professor Struthers of Aberdeen. Without their kind aid this part of my investigation could not have been carried out, because in the Museum of Trinity College, Dublin, the number of spines of low races is very limited, and I had necessarily to ask the permission of these gentlemen to examine the specimens under their charge. This was accorded to me in the freest manner possible. More especially, however, I feel indebted to Dr. Garson and Professor Macalister, both of whom gave me very material help in making the measurements, and from whom I received many suggestions, which have proved of the utmost value to me.

Table G.

INDICES OF LUMBAR VERTEBRÆ IN SEVENTEEN AUSTRALIAN SPINES.

	10 Males.										Average.
I. Lumbar Vertebra, .	113·6	119·0	106·8	128·7	110·8	135·1	130·2	153·3	126·0	128·2	**125·1**
II. Lumbar Vertebra, .	120·0	111·3	111·3	116·3	112·7	125·6	115·2	132·3	113·3	108·7	**116·6**
III. Lumbar Vertebra, .	114·3	108·7	123·8	104·3	102·0	122·7	120·4	138·7	122·0	108·7	**116·5**
IV. Lumbar Vertebra, .	95·6	98·0	102·0	102·1	93·8	109·3	106·3	136·3	104·4	97·0	**104·4**
V. Lumbar Vertebra, .	86·6	91·5	95·6	71·7	81·6	97·6	100·0	103·1	88·8	84·0	**90·0**
Lumbo-vertebral Index,	105·4	105·2	107·5	104·0	100·0	117·6	114·2	132·5	110·5	104·3	**110·1**

	4 Females.				Average.	3 Sex unknown.			Average. 17. Both Sexes.
I. Lumbar Vertebra, . .	107·0	102·0	125·0	111·9	111·4	111·6	108·7	—	**119·8**
II. Lumbar Vertebra, . .	106·6	98·2	104·3	107·3	104·1	113·9	113·0	111·5	**113·0**
III. Lumbar Vertebra, . .	100·0	108·7	109·0	117·5	108·2	113·6	108·7	108·3	**113·6**
IV. Lumbar Vertebra, . .	97·8	89·6	104·5	110·0	100·4	107·1	109·0	—	**103·9**
V. Lumbar Vertebra, . .	92·8	86·0	87·5	102·3	92·1	87·0	91·2	—	**90·4**
Lumbo-vertebral Index, .	100·9	96·7	105·4	109·7	103·1	106·4	106·0	—	**107·8**

Table H.

INDICES OF LUMBAR VERTEBRÆ IN THREE TASMANIAN SPINES.

	2 Males.		Average.	1 Female.	Average. 3. Both Sexes.
I. Lumbar Vertebra, . .	111·6	124·4	118·0	109·5	**115·1**
II. Lumbar Vertebra, . .	111·3	111·3	111·3	107·1	**109·9**
III. Lumbar Vertebra, . .	108·7	114·6	111·6	107·1	**110·1**
IV. Lumbar Vertebra, . .	108·8	112·2	110·5	107·5	**109·5**
V. Lumbar Vertebra, . .	91·2	93·1	92·1	93·0	**92·4**
Lumbo-vertebral Index, . .	106·2	110·9	108·5	104·7	**107·2**

Table I.

INDICES OF LUMBAR VERTEBRÆ IN TWENTY-THREE ANDAMAN SPINES.

	14 Males.														Average.
I. Lumbar Vertebra,	107·5	119·5	116·3	106·8	116·2	127·7	117·3	111·1	113·6	113·3	113·6	113·3	117·2	109·0	**114·4**
II. Lumbar Vertebra,	107·1	113·6	113·6	107·0	106·8	144·1	110·2	113·3	115·9	110·8	106·3	118·1	117·2	106·6	**113·5**
III. Lumbar Vertebra,	112·8	102·0	113·6	102·2	109·0	128·9	108·3	117·5	111·3	113·0	106·2	104·1	117·0	106·6	**110·9**
IV. Lumbar Vertebra,	110·0	95·6	98·0	97·6	98·0	113·5	106·0	107·0	104·1	104·3	102·0	104·3	106·0	106·6	**103·7**
V. Lumbar Vertebra,	97·0	80·4	91·6	83·0	95·3	89·5	94·2	95·5	93·6	87·5	88·2	104·5	82·6	88·0	**90·7**
Lumbo-vertebral Index,	107·0	101·8	106·1	99·0	105·0	118·8	106·9	108·6	107·4	105·6	102·8	108·8	107·6	103·0	**106·2**

	9 Females.									Average.	Average. 23 Both Sexes.
I. Lumbar Vertebra, .	116·3	116·6	112·2	108·8	100·0	113·9	111·1	111·6	97·7	109·8	**112·6**
II. Lumbar Vertebra, .	104·3	116·3	100·0	108·8	101·2	111·1	111·1	114·2	100·0	107·3	**111·2**
III. Lumbar Vertebra, .	98·3	109·3	100·0	—	96·0	108·7	106·3	111·6	95·9	104·2	**108·1**
IV. Lumbar Vertebra, .	98·0	113·6	91·3	102·2	98·0	95·9	109·0	102·2	97·9	100·9	**102·6**
V. Lumbar Vertebra, .	82·0	91·5	85·1	97·6	87·0	93·3	102·3	100·0	94·0	92·5	**91·4**
Lumbo-vertebral Index,	99·0	109·1	97·3	—	96·6	104·3	108·0	107·9	97·0	102·4	**104·8**

Table J.

INDICES OF LUMBAR VERTEBRÆ IN TEN NEGRO SPINES.

	7 Males.							Average.	3 Females.			Average.	Average. 10 Both Sexes.
I. Lumbar Vertebra, .	112·0	116·6	111·9	113·0	125·0	119·0	[1] 107·1	114·9	107·1	110·5	113·6	110·4	**113·5**
II. Lumbar Vertebra, .	120·0	116·6	113·9	106·1	104·7	117·3	109·0	112·5	115·0	110·5	100·0	108·5	**111·3**
III. Lumbar Vertebra, .	108·9	118·1	116·6	106·0	91·4	94·2	109·2	106·3	107·0	119·4	88·2	104·8	**105·9**
IV. Lumbar Vertebra, .	101·7	114·2	107·0	98·0	104·4	100·0	100·0	103·4	104·7	117·6	104·1	108·8	**105·1**
V. Lumbar Vertebra, .	93·1	97·2	100·0	100·0	97·7	84·6	86·1	94·1	95·4	100·0	65·4	86·9	**92·0**
Lumbo-vertebral Index, .	106·6	112·6	109·9	104·5	103·7	102·0	102·2	106·0	105·6	111·3	93·4	103·4	**105·4**

[1] Unusually tall male skeleton (about 6 ft. 3 in.), presented to the Author by Dr. F. Alcock Nixon, of the Ledwich School of Medicine, Dublin.

Table K

INDICES OF LUMBAR VERTEBRÆ IN THREE BUSHMEN.

	1 Male.	2 Females.		Average.	Average. 3 Both Sexes.
I. Lumbar Vertebra, . . .	109·0	115·9	123·0	119·4	**115·9**
II. Lumbar Vertebra, . . .	120·0	110·8	109·5	110·1	**113·4**
III. Lumbar Vertebra, . . .	102·0	108·3	119·5	113·9	**109·9**
IV. Lumbar Vertebra, . . .	93·7	104·1	104·7	104·4	**100·8**
V. Lumbar Vertebra, . . .	90·4	98·0	97·7	97·8	**95·3**
Lumbo-vertebral Index, . .	102·1	107·2	110·5	108·8	**106·6**

The indices obtained in the case of the Andamans and Negroes exhibit a striking similarity, and this is a point of considerable interest in the light of the view which has recently been so ably advocated by Professor Flower regarding the affinities which exist between these races. He looks upon the Andaman as being the representative of a primitive undeveloped, infantile type, from which both the African and Oceanic Negro have sprung.[1]

A similar correspondence between the Australian and Tasmanian indices will be noted; but as only three spines of the latter race were examined, we cannot lay much stress upon this similarity.

The condition of the separate vertebræ, in the five low races under consideration, can be most easily understood and compared with the condition of the European vertebræ by arranging them in the following tabular form:—

[1] Journal of the Anthropological Institute, 1879.

Table L.

FIFTY-SIX SPINES OF THE LOWER RACES OF MAN.

	Anterior depth greater than posterior depth.	Anterior depth equal to posterior depth.	Anterior depth less than posterior depth.
I. Lumbar Vertebra, .	1	1	54
II. Lumbar Vertebra, .	1	3	52
III. Lumbar Vertebra, .	6	2	48
IV. Lumbar Vertebra, .	17	2	37
V. Lumbar Vertebra, .	47	5	4

The wide difference between the results brought out by this Table, and that constructed upon a similar plan for the Europeans (Table F, p. 12), is very evident. More particularly does the condition of the last lumbar vertebra attract our attention. In five cases (viz. in one Andaman, one Australian, and three Negroes) it presents an equal depth in front and behind, and in four cases (viz. in two Andamans and two Australians) its posterior depth exceeds its anterior depth. In one instance, indeed, that of a male Andaman, the index of this vertebra was 104·5. In the European spines the highest index observed for the last lumbar vertebra was 94·5, and this was in one of the few examples in which the general lumbo-vertebral index was above the standard (104·1).

In only seven out of the fifty-six low-race spines which were measured was the general lumbo-vertebral index below the standard. These included five Andamans, one Australian, and one Negro; and it is significant to note that, of these exceptions to the general rule, six were the indices of females, and only one the index of a male. Moreover, the latter was very slightly below the standard, viz. 99.

Anthropoid Apes.—The following are the detailed results which have been obtained in the case of the four anthropoid, and also of several of the lower apes:—

Table M.—INDICES OF LUMBAR VERTEBRÆ IN THE GORILLA.

	3 Males.			Average.	3 Females.			Average.	Average. 5 Both Sexes.
XIII. Dorsal Vertebra, ...	131·8	106·1	103·8	113·9	125·0	110·7	115·0	116·9	**115·3**
I. Lumbar Vertebra, ...	109·2	106·4	105·4	107·0	114·3	110·0	125·0	116·4	**111·7**
II. Lumbar Vertebra, ...	107·2	110·3	105·3	107·6	114·0	109·3	121·9	115·0	**111·3**
III. Lumbar Vertebra, ...	101·7	104·8	103·3	103·2	100·0	115·6	106·5	107·3	**105·3**
IV. Lumbar Vertebra, ...	94·7	111·5	103·3	103·1	fused to sacrum.	107·9	92·0	99·9	**101·9**
Lumbo-vertebral Index, ...	107·7	107·7	104·2	106·5	—	110·0	111·0	110·5	**108·1**

Table N.—INDICES OF LUMBAR VERTEBRÆ IN THE CHIMPANZEE.

	3 Males.			Average.	5 Females.					Average.	Young. Sex doubtful.	Average. 9 Both Sexes.
XIII. Dorsal Vertebra, .	125·0	107·0	125·7	119·2	150·0	120·0	110·8	116·6	—	124·3	147·6	**125·3**
I. Lumbar Vertebra, .	135·3	113·6	112·5	120·4	105·0	102·2	118·9	119·2	122·2	113·5	125·0	**117·1**
II. Lumbar Vertebra, .	130·5	132·5	111·6	124·8	104·3	118·6	112·5	107·7	106·5	109·9	124·0	**116·4**
III. Lumbar Vertebra, .	120·0	139·0	111·9	123·6	114·2	102·3	100·0	119·2	111·3	109·6	126·0	**116·1**
IV. Lumbar Vertebra, .	125·0	125·6	107·8	119·4	111·6	107·0	120·0[1]	113·4	121·4	114·6	111·1	**115·8**
Lumbo-vertebral Index, .	127·1	123·0	113·5	121·2	114·2	109·6	111·9	115·2	—	112·7	125·8	**117·5**

[1] This vertebra was fused to sacrum, but could readily be measured.

Table O.—INDICES OF THE LUMBAR VERTEBRÆ IN THE ORANG.

	3 Males.			Average.	1 Female.	Average. 4 Both Sexes.
XII. Dorsal Vertebra, ...	109·3[1]	111·1[1]	115·9	112·1	118·5	**113·7**
I. Lumbar Vertebra, ...	100·0	129·4	112·0	113·8	134·4	**118·9**
II. Lumbar Vertebra, ...	102·0	151·6	105·3	119·6	120·0	**119·7**
III. Lumbar Vertebra, ...	86·2	126·3	103·3	105·2	132·1	**111·9**
IV. Lumbar Vertebra, ...	96·4	105·0	92·7	98·0	120·0	**103·5**
Lumbo-vertebral Index, ...	98·0	123·4	105·3	108·9	125·0	**112·9**

[1] In these specimens there were five lumbar vertebræ.

Table P.

INDICES OF LUMBAR VERTEBRÆ OF THE GIBBON.

	Hylobates lar.	H. lar. (Aeby.)[1]	H. lar.	H. variegatus.	H. leuciscus.	H. syndactylus. (Aeby.)[1]	Average. 6 Both Sexes.
I. Lumbar Vertebra, . . .	107·1	115·3	103·4	121·7	117·3	112·5	**112·8**
II. Lumbar Vertebra, . . .	113·3	115·3	103·3	107·4	108·0	105·8	**108·8**
III. Lumbar Vertebra, . . .	112·5	107·1	100·0	111·5	111·5	102·8	**107·5**
IV. Lumbar Vertebra, . . .	105·8	107·4	106·4	103·8	115·3	100·0	**106·4**
V. Lumbar Vertebra, . . .	106·6	100·0	96·2	96·0	129·1	97·2	**104·1**
Lumbo-vertebral Index, . . .	109·0	109·0	102·0	107·8	116·1	103·4	**107·1**

[1] Indices calculated from measurements given by Professor Aeby in his "Beiträge zur Osteologie des Gorilla." Morph. Jahr., vol. iv.

Table Q.

INDICES IN LUMBAR VERTEBRÆ IN SOME OF THE LOWER APES.

		Papio sphinx. (Aeby.)[1]	Cynocephalus anubis.	Cynocephalus anubis.	Macacus nemestrinus. (Aeby.)[1]	Macacus nemestrinus.	Macacus nemestrinus.	Colobus velerosus.	Cercopithecus. (Aeby.)[1]	Semnopithicus entellus.
Lower five true Vertebræ.	*a*	103·0	128·9	106·6	117·6	108·1	103·5	103·8	116·6	108·1
	b	100·0	130·7	110·6	110·5	102·4	110·7	103·8	115·3	112·7
	c	100·0	116·0	100·0	100·0	102·2	106·8	103·8	115·3	112·2
	d	95·0	108·5	106·1	100·0	102·3	107·4	108·3	115·3	102·4
	e	77·7	89·7	96·0	86·3	102·7	100·0	90·4	100·0	89·2
Lumbo-vertebral Index,		95·1	113·4	103·7	101·9	103·4	105·9	102·4	112·5	105·1

[1] Indices calculated from measurements given by Professor Aeby in his "Beiträge zur Osteologie des Gorilla."

The condition of the last lumbar vertebra in the anthropoid apes naturally attracts our attention first. The low index of this vertebra in man has been seen to be a point of primary importance. Is the high average index of the corresponding vertebra of the Anthropoid borne out by the indices of the different individuals? In the chimpanzee it is; in every case the index of this vertebra is above the standard: but in two gorillas, two orangs, and three gibbons it is distinctly below the standard, and therefore approaches the human type. In this respect, therefore, a hard and fast line cannot be drawn between the lower races of man and the anthropoid: a certain amount of overlapping in each case takes place.

A study of the separate indices of the individual anthropoids reveals a striking want of uniformity in the condition of the vertebræ. Whilst the average indices exhibit a regular decrease from above downwards, no such principle can be deduced from the examination of the indices of each individual. This is, in a marked degree, different from the separate indices of the European, and to a less extent, also, it is different from the separate indices of the lower races of man.

Sexual Differences.—It would appear from the observations of Luschka, Balandin, Charpy, and Ravenel, that the lumbar curve is more marked in the human female than in the male. In the second volume of his well-known treatise upon "Human Anatomy," Luschka states that "in the female sex the lumbar column is not only somewhat higher than in the male, but it is also more strongly bent;" and he further remarks that a strong loin-excavation on the posterior aspect of the trunk is a mark of a well-formed female body. He omits, however, to mention by what means he arrived at these conclusions. Balandin,[1] in his investigations into the physiological curves of the human spine, found in his preparations "that the male vertebral columns show a smaller degree of curvature, but a greater stability, than the female." He examined a very

[1] "Beiträge über die Entstehung der physiologischen Krümmung der Wirbelsaüle beim Menschen," von Dr. J. Balandin in St. Petersburg. Virchow's Archiv für. Path. Anat. und Phys. 1873.

large number of specimens, but his methods were of such a kind that absolute accuracy could not be expected. He removed from the body that section of the column which is composed of all the true vertebræ, and, leaving the thoracic walls attached, he stripped the specimen of all the soft parts, with the exception of the ligaments. The spine, thus prepared, was then placed on its side upon a table, and the curves determined by means of the profile view. The stability of the column was estimated by the resistance it offered to stretching when grasped by the hands at either end. Dr. Adrien Charpy[1] of Lyons has recently gone more fully into the question. He points out that the degree of lumbar curve is, as a rule, in accordance with the inclination of the sacrum, and that in the two sexes, at a corresponding age, the sacro-vertebral angle is slightly more acute in the female than in the male (viz. 110° in the male, and 107° in the female)—"This same difference is also found in new-born children, which proves that it is a congenital condition, transmissible by heredity." He has further directly estimated the amount of lumbar curve in twenty-five young well-formed subjects by moulding a strip of lead upon the anterior face of the column. The average results gave a superior curvature to the female.

Ravenel[2] has approached the question from a different standpoint. By measurements of the anterior and posterior surfaces of the column formed by the vertebral bodies and the intervening cartilaginous discs in eleven males and an equal number of females, he has arrived at the following conclusions:—(1) That the anterior face of the lumbar segment is in a small degree relatively longer in the female than in the male; and (2) that the posterior surface of the same segment is relatively very considerably shorter in the female than in the male. Taking the fore surface of this section of the column as equal to 100 in both sexes, he estimates that the posterior surface in the male is 86·6; whilst in the female it is only 69·6. If this startling difference

[1] "De la Courbure Lombaire et de l'inclinaison du bassin. Journal de l'Anatomie et de la Physiologie." (Robin et Pouchet.) Paris, 1885. No. 4.

[2] "Die Maasverhältnisse der Wirbelsaüle und des Rückenmarkes beim Menschen," von Michel Ravenel. Zeitschrift für Anatomie und Entwicklungsgeschichte. 1877.

can be substantiated, it will undoubtedly constitute, as Ravenel claims, "ein spezifischer Geschlechtsunterschied." He includes the intervertebral disc between the last lumbar vertebra and the sacrum in his measurements. This is strongly influenced in its shape by the inclination of the pelvis, and according to Charpy the sacro-vertebral angle is, as a rule, more acute in the female than in the male. Differences resulting from this, however, would be of small moment. Much more likely were inaccuracies and fallacies to creep in from the methods which he adopted in manipulating the vertebral column previous to and during the measurements. He took no precaution towards the fixing of the vertebral bodies in their mutual relationships, but straightway exposed the two faces of the column by the removal of the neural arches and spinal cord from its posterior aspect, and the viscera from its anterior aspect. Hirschfeld[1] rightly insists upon the influence which the powerful elastic ligaments connecting the neural arches exert upon the maintenance of the curves; and everyone who has tried the experiment knows that, when these are removed, the curves of the column diminish in degree, whilst the flexibility increases; indeed, it can now be bent with facility in every direction. But further, in making his measurements upon the spines thus mutilated, Ravenel found it necessary to support the curves by means of props.

The results which I have arrived at from measurements of the entire length of the lumbar region in the two sexes does not agree with those obtained by Ravenel. It is right to bear in mind, however, that although I claim a greater exactitude of method, the number of individuals I have examined is small in comparison with his. I have frozen the spines of five well-formed males, and four well-formed females, and then divided them in the mesial plane by means of a saw. While still in the frozen condition, with each element immovably fixed in proper position, I have taken careful tracings of the cut surface. In a reduced form these are figured in Plate II. (Nos. 1, 3, 4, 5, 6, males, and Nos. 1, 2, 3, 4, females). To this series I

[1] "Nouvel aperçu sur les conditions anatomiques des courbures de la colonne vertébrale chez l'homme." Par M. L. Hirschfeld. Gaz. Méd. de Paris. 1849.

have added tracings of Braune's well-known plates of the male and female in mesial section [Nos. 2 (male) and 5 (female)], and of the male spine figured by the Brothers Weber.[1] The accuracy of the last is sufficiently attested by the fact that Parow,[2] in seeking a means of determining the curves in the living individual, chose it as one of his standards.

Upon the tracings thus obtained, the anterior and posterior face of the lumbar region in each case could be measured with the greatest accuracy. In the first instance I adopted Ravenel's delimitation of the lumbar section, and measured each face from the level of the upper surface of the first lumbar vertebra to the level of the upper surface of the sacrum. In other words the lumbo-sacral intervertebral disc was included in the region. In comparing the results obtained in the two sexes I was surprised to find no essential difference. Taking the fore surface as the standard, and equal to 100, the following were the indices :—

Five females, ...	85·3
Seven males, ...	85·6

These indices stand in striking contrast to those given by Ravenel.

I next eliminated the lumbo-sacral intervertebral disc, and measured both aspects of the column from the level of the upper surface of the first lumbar vertebra to the level of the lower surface of the last lumbar vertebra. This gave slightly different results, and showed in the female a greater length of the anterior face, in proportion to the posterior face; in short, it indicated a stronger curve in the female. The average indices were :—

Five females, ...	88·1
Seven males, ...	89·1

The results obtained from the two sets of measurements, therefore, turned out to be the very reverse of what I had anticipated, and

[1] *Antea*, p. 3.

[2] "The Physical Conditions of the Upright Position and Curvatures of the Spine." Virchow's Archiv. Band xxxi., p. 108.

to reconcile the discrepancy it was necessary to determine the sacro-vertebral angle in each case. This can be done in a variety of ways, but the most convenient, and it seems to me the most accurate method, is to prolong the lines joining the central points of the upper and lower

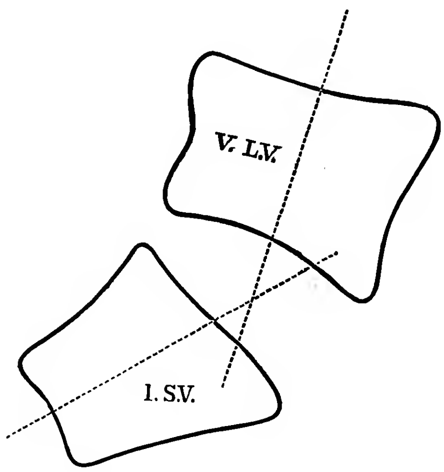


Fig. 1.

surfaces of the last lumbar and first sacral vertebræ respectively (in other words, to prolong the axis-line of each of these vertebræ). These lines intersect in the lumbo-sacral disc of cartilage; and in the twelve spines figured in Plate II., the average angle for the males was 133° 6′, and for the females 137° 40′. The lumbo-sacral angle, therefore, is more acute, and the promontory more marked in the males than in the females; and this accounts for the lower index which was obtained for the males when the lumbo-sacral disc of cartilage was included in the measurements. The following Table gives the detailed results in the case of each spine:—

Table R.

		Seven Males figured in Plate II.								Five Females figured in Plate II.					
		No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.	Weber's	Average.	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	Average.
Relative height of fore surface of Lumbar region in comparison with the relative height of back surface: taking the fore surface as 100.	Includes Lumbo-sacral cartilage.	86·7	82·5	88·1	88·6	85·7	82·1	85·4	85·6	88·0	87·4	85·7	82·2	83·5	85·3
	Exclusive of Lumbo-sacral cartilage.	91·5	86·3	90·5	91·5	91·0	84·2	89·0	89·1	90·0	91·6	88·0	86·9	84·0	88·1
Lumbo-sacral angle,		137° 0′	130° 30′	135° 0′	125° 30′	138° 45′	138° 30′	126° 30′	133° 6′	147° 0′	137° 30′	117° 20′	136° 0′	150° 30′	137° 40′
Curve Index,		10·3	10·3	6·9	5·4	8·9	12·9	7·1	8·8	8·0	6·0	9·7	12·0	11·7	9·5
Indices of Vertebral bodies.	I. Lumbar Vertebra,	113·2	103·9	112·0	105·6	103·7	106·0	107·6	. .	104·0	104·0	97·9	100·0	100·0	
	II. Lumbar Vertebra,	100·0	96·5	103·7	101·7	105·3	103·8	107·4	. .	100·0	96·2	100·0	100·0	101·9	
	III. Lumbar Vertebra,	89·8	88·3	105·8	98·3	100·0	94·4	98·2	. .	98·1	96·2	92·0	91·2	92·3	
	IV. Lumbar Vertebra,	87·7	93·4	100·0	96·6	93·3	86·2	91·2	. .	94·2	98·0	88·4	90·9	80·3	
	V. Lumbar Vertebra,	79·3	67·7	85·7	80·0	90·0	68·9	80·0	. .	82·6	75·4	76·9	80·3	81·4	
Lumbo-vertebral Index,		93·6	89·3	101·1	96·1	98·0	91·1	96·4	. .	95·7	93·8	90·8	92·3	91·0	

Note.—The fact that the average lumbo-sacral angle in the twelve spines examined by me is more acute in the males than in the females cannot be regarded as in any way militating against the general results obtained by Charpy.[1] In his investigations he determined the angle in the spines of subjects of the same age; in my specimens the sum of the ages of the males was slightly greater than the sum of the ages of the females.

[1] *Antea*, p. 23.

But there is another and more simple method by which the degree of lumbar curvature in the tracings of the different spines may be compared, viz. by drawing a straight line from the anterior border of the lower surface of the last lumbar vertebra to the anterior border of the upper surface of the first lumbar vertebra. By the eye we can readily judge the amount of projection which takes place in front of that line in the different tracings (Plate II.), but for accurate comparison it is well to formulate an index. This can readily be done by taking the length of the lumbar region (measured from the centre of the upper surface of the first lumbar vertebra to the centre of the lower surface of the fifth lumbar vertebra) as the standard, and equivalent to 100, and then comparing it with the distance between the intersecting line and the point of greatest prominence. The most projecting point will generally be found to be the anterior border of the upper surface of the fourth lumbar vertebra in the male, and the anterior border of the lower surface of the third lumbar vertebra in the female. The index may be calculated thus:—

$$\frac{\text{Distance of most prominent point in front of intersecting line} \times 100}{\text{Length of lumbar column.}}$$

A high index will, therefore, indicate a strongly pronounced curve, and a low index a feeble degree of curvature; it may be termed the *index of the lumbar curve.*

In Table R, p. 27, the *curve indices* for each spine, figured in Plate II., is given, and the *average curve index* for the two sexes will be seen to be:—

Seven males, ...	8·8
Five females, ...	9·5

In every respect these data bear out the results obtained by measurement of the two faces of the lumbar column after excluding the lumbo-sacral intervertebral disc.

The evidence, therefore, which is obtained by an examination of the lumbar part of the vertebral column, as a whole, is entirely in favour of

the view that the curvature is more pronounced in the female than in the male; and we may now turn to the individual vertebræ and see what information, bearing upon this point, may be gathered from an examination of their centra in the two sexes. As will be seen from the following Table, the results obtained are of a very important character:—

TABLE S.

	IRISH.		ANDAMANS.		NEGROES.		AUSTRALIANS.		TASMANIANS.		BUSHMEN.	
	21 ♂	22 ♀	14 ♂	9 ♀	7 ♂	3 ♀	10 ♂	4 ♀	2 ♂	1 ♀	1 ♂	2 ♀
I. Lumbar Vertebra, .	106·9	102·6	114·4	109·8	114·9	110·4	125·1	111·4	118·0	109·5	109·0	119·4
II. Lumbar Vertebra, .	102·0	98·1	113·5	107·3	112·5	108·5	116·6	104·1	111·3	107·1	120·0	110·1
III. Lumbar Vertebra, .	97·7	95·0	110·9	104·2	106·3	104·8	116·5	108·2	111·6	107·1	102·0	113·9
IV. Lumbar Vertebra, .	95·0	91·8	103·7	100·9	103·4	108·8	104·4	100·4	110·5	107·5	93·7	104·4
V. Lumbar Vertebra, .	82·4	81·2	90·7	92·5	94·1	86·9	90·0	92·1	92·1	93·0	90·4	97·8
Lumbo-vertebral Index, .	96·2	93·5	106·3	102·4	106·0	103·4	110·1	103·1	108·5	104·7	102·1	108·8

Of the forty-three Irish spines examined, twenty-one were males, and twenty-two females. The *lumbo-vertebral index* of the latter is distinctly lower than that of the former, viz.:—

Females, ... 93·5
Males, ... 96·2

The vertebral bodies of the female are thus moulded more in adaptation to the curve than those of the male; and here again, therefore, we have corroborative proof as to the superiority of the female lumbar curve. Aeby[1] was unable to find any difference in this respect between the sexes, but it is evident that his observations were on too limited a scale. It will

[1] "Der Bau des menschlichen Körpers."

be further noticed in Table S, that in the Irish female the only lumbar vertebra with an average index above the standard is the first member of the series.

But this sex distinction is not limited to the European; the same difference between the lumbo-vertebral indices of the male and female is to be noted in every race examined, with the exception of the Bushmen, and of these only three spines were measured. Charts C and D (Plate I.) will render these several differences more apparent. In chart C all the races (with the exception of the Bushmen) are included; the females are indicated by dotted lines, and the males by solid lines. In chart D the Tasmanians and Negroes are omitted in order to simplify the diagram. Except in the column which is allotted to the last lumbar vertebra, the female lines will all be noticed to occupy a lower level than the corresponding male lines. The European lines proceed together, apart from the others, and throughout they occupy the lowest place on the chart.

In the anthropoid apes we see no such sex distinction, except, perhaps, in the chimpanzee; and in view of the great irregularity exhibited by the lumbo-vertebral index in the individual members of this group, we cannot attach any importance to this discrepancy between the sexes:—

TABLE T.

		Gorilla.		Chimpanzee.		Orang.	
		3 ♂	3 ♀	3 ♂	5 ♀	3 ♂	1 ♀
Lower five true Vertebræ.	*a*	113·9	116·9	119·2	124·3	112·1	118·5
	b	107·0	116·4	120·4	113·5	113·8	134·4
	c	107·6	115·0	124·8	109·9	119·6	120·0
	d	103·2	107·3	123·6	109·6	105·2	132·1
	e	103·1	99·9	119·4	114·6	98·0	120·0
Lumbo-vertebral Index, .		106·5	110·5	121·2	112·7	108·9	125·0

In the case of the gorilla and orang the lumbo-vertebral index exhibits an opposite condition, but the number of individuals examined has been too small to justify us in formulating any law upon this point.

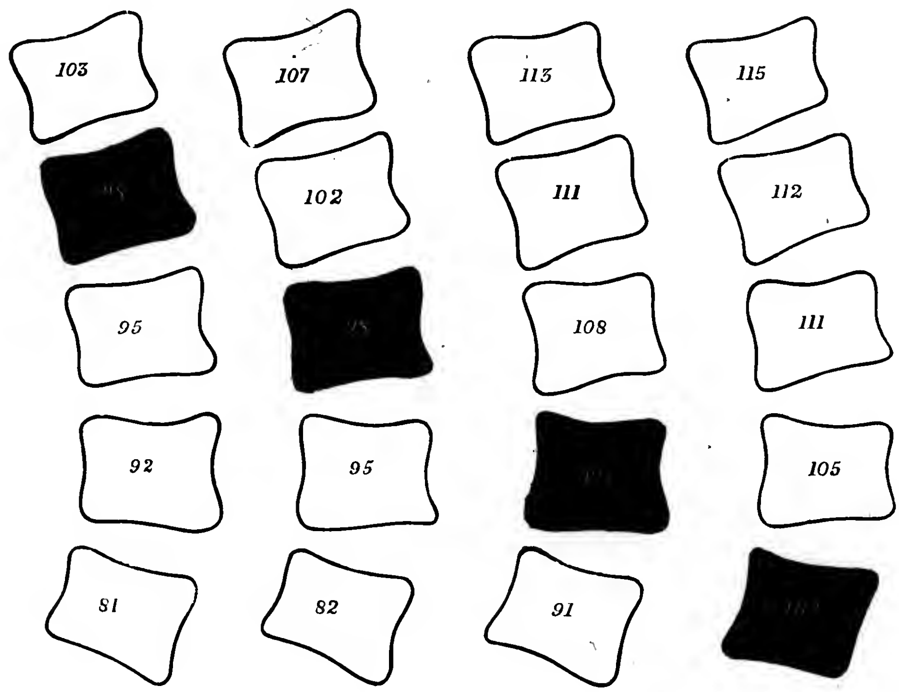
Dr. Charpy in his Paper upon the lumbar curve, to which we have already had occasion more than once to refer, discusses the influences at work in producing a more prominent curve in the human female than in the male; and the views which he advances are very reasonable. He says:—

"Si la courbure lombaire est normalement, originellement plus forte chez la femme, à plus forte raison cette différence va-t-elle en s'accentuant, dans le cours de la vie, sous l'influence de circonstances en quelque sorte aggravantes et spéciales au sexe féminin. En première ligne se place la grossesse, qui oblige la colonne dorsale à se reporter en arrière et impose aux muscles lombaires extenseurs et incurvateurs un effort proportionnel au poids surajouté du côté de la flexion. Et ce n'est pas seulement pendant quelques mois que cet effet se manifeste; souvent toute la vie persiste un affaiblissement de la paroi abdominale qui laisse prédominer les muscles redresseurs antagonistes. Il est naturel de croire que c'est l'attitude de la grossesse, qui, poursuivie dans une série incalculable de générations a fini par créer un type héréditaire et originel, indiqué vaguement dès le premier âge, nettement accusé à la puberté et recevant tout son développment par les fonctions maternelles de la gestation et du port de l'enfant allaité."

It seems to me that the carrying of the child either in the arms or upon the back must have a more important influence in determining the greater curvature of the spine than the occurrence of intermittent pregnancies, or the slight disturbance which results in the balance of power between the extensors and flexors of the spine. At all events it is clear that it is a hereditary condition.

Which Vertebra in Man and the Anthropoids shows an Index most nearly approaching the Standard?—In comparing the average indices of the sepa-

rate vertebræ, with a view to determine this question, a somewhat interesting point becomes apparent, as will be seen from the accompanying diagram :—



European. ♀ European. ♂ Andaman. Gorilla.

Fig. 2.

The numbers on the different vertebral bodies indicate the average index of that vertebra in the four groups represented in the diagram. The vertebral body in black shading is that member of the lumbar series which possesses an index most nearly approaching the standard; in other words, it is the vertebra of transition.

In the anthropoid ape the vertebra which shows an index most nearly approaching the standard is the last lumbar; in the low races it is the fourth lumbar; in the European male it is the third lumbar;[1] and in the European female it is the second lumbar. It ascends, therefore, step by step, from the anthropoid ape to the European female. But in constructing the accompanying diagram, another curious point became evident, viz. that (with the exception of the two lower lumbar vertebræ) the index of a given vertebra in the European female corresponds closely with the index of the vertebra in the male immediately below it. A similar correspondence will be noted in the case of the Andaman and the gorilla, but not in so marked a degree.

Relation of the form-adaptation of the Vertebral Bodies to the degree of Lumbar Curvature.—An important point must now be discussed: Has the moulding of the vertebral bodies any relation to the degree of lumbar curvature? We have noted that the lumbo-vertebral index, which expresses the amount of this form-adaptation, is very variable in different individuals; the question, therefore, comes to be: Is a low lumbo-vertebral index associated with a high degree of curvature, and *vice versa?* This point can in a measure be determined by comparing in Table R, p. 27, the lumbo-vertebral index of each spine figured in Plate II., with the corresponding index of curve. In so far as the individual spines are concerned, it will be noticed that there is a general correspondence between the two indices. But this can be brought out in a more striking manner if we calculate the average indices for the six spines which exhibit the highest degree of curvature in the lumbar region, and compare these with the

[1] In the European male it is the second lumbar vertebra which, strictly speaking, most nearly approaches the standard index, viz. 102, as against 97·7 for the third lumbar vertebra; but in view of the other relationships mentioned above, I have thought myself justified in overlooking this minute difference, and in giving the preference to the vertebra which possessed an index below the standard. The Andamans were chosen for comparison, because I had examined a greater number of specimens from this race than in the case of the other low races; and the gorilla, on account of the closer correspondence of its indices to those of man, than in the case of the other anthropoids.

average indices obtained from the six spines which exhibit the lowest degree of curvature. The following is the result:—

	Average Index of Lumbar Curvature.	Average Lumbo-vertebral Index.
The six spines in Plate II. which exhibit the lowest degree of curve in the lumbar region, viz., Nos. 3, 4, and 5, males, Nos. 1 and 2, Females, and Weber's figure, ...	7·0	96·8
The six spines in Plate II. which show the highest degree of lumbar curve, viz., Nos. 1, 2, and 6, Males, and Nos. 3, 4, and 5, Females, . .	11·1	91·3

It is clear, therefore, that the bodies of the vertebræ in the lumbar region are fashioned in a more or less marked manner in accordance with the degree of the lumbar curvature. In all cases, however, the difference in height between the anterior and posterior surfaces of the lumbar vertebræ (with the exception of the fifth) is so slight that it can have little or no influence in determining the curve. Weber,[1] Nühn,[2] Meyer,[3] Hirschfeld,[4] &c., have all given expression to a similar opinion. As Meyer,[5] Horner,[6] Aeby,[7] and Pansch[8] remark, the difference in height between the anterior and posterior surfaces of the lumbar vertebral bodies must be looked upon as *the consequence*, and not as *a cause* of the curve. But whilst this is the case, the condition which produces it is not, as some suppose, the unequal pressure to which the centrum is subjected during the life of the individual. Its cause is not immediate and mechanical. If it were so we would find it equally developed in all races; whereas in the low races the fifth lumbar vertebra alone exhibits the peculiarity—and this in a minor degree. Who, on such scanty evidence, would presume to say that in these races the lumbar curve is less marked than in the European? The condition is a hereditary one, and it has originated from influences operating upon the bodies of the vertebræ, as the lumbar curve has become in successive generations more and more firmly established. At the same time it must

[1] *Antea*, p. 3. [2] *Antea*, p. 3. [3] *Antea*, p. 3. [4] *Antea*, p. 3.
[5] *Antea*, p. 3. [6] *Antea*, p. 3. [7] *Antea*, p. 3. [8] *Antea*, p. 3.

be admitted that it is very doubtful if the new-born child exhibits a like condition of the bodies of the lumbar vertebræ. At this early age, when only the central parts are ossified, it is a matter of extreme difficulty to measure the vertebral bodies accurately. Bouland[1] found a greater height of the lumbar column in front than behind, and that as a rule the osseous nuclei, the cartilages of ossification, and the intervertebral discs, all concurred to produce this. Horner[2] and Aeby,[3] however, believe that in the first instance the bodies of the lumbar vertebræ are equally deep in front and behind. From a series of careful measurements which I have made myself, I am inclined to think that the difference between the anterior and posterior vertical diameters of the lumbar vertebræ in the new-born child (if indeed there is any) is very slight indeed, and that it is sometimes in favour of the front surface, and sometimes in favour of the back surface of the column.

From what has been said, some might be inclined to argue that the European had assumed the erect attitude at a period antecedent to the low races. Such a deduction would be altogether untenable. The difference in the form-adaptation of the lumbar bodies with reference to the curve, in a European and in a low race, is easily explained when we reflect upon the difference in their habits. The European who leads a life which rarely necessitates his forsaking the erect attitude, except as an intermittent occurrence, and then for short periods, has sacrificed in the lumbar part of the vertebral column *flexibility* for *stability*. It is evident that the deeper the bodies of the vertebræ grow in front, the more permanent, stable, and fixed the lumbar curve will become, and the more restricted will be the power of forward bending at this region of the spine. The savage, in whose daily life agility and suppleness of body are of so great an account, who is frequently called upon to pursue game in a prone position, and climb trees in search of fruit, preserves the pithecoid condition of vertebræ in the lumbar region; and on account of this a superior flexibility of the spine must result. As we have stated, there is no reason to suppose that this condition is associated with a smaller degree of curvature in this region.

[1] "Journal de l'Anatomie et Physiologie," Paris, 1872.

[2] *Antea*, p. 3.

[3] Der Bau des menschlichen Körpers.

Indeed it is well known that many of the lower races are distinguished by their erect and graceful carriage, and by the depth of the loin excavation on the posterior aspect of the trunk—an appearance which would indicate a strong degree of lumbar curvature.

The following Table gives the indices of the lumbar vertebræ in the Chinese, Esquimaux, and certain natives of India. As the number of spines belonging to each of those races which I had an opportunity of examining was very small, I have not included the results in the foregoing Tables:—

TABLE U.

	3 Esquimaux (2 males and 1 female).	1 native of Canton (male).	2 Hindu males.	1 native of Bhutan (female).	1 native of Maldive Islands (male).	2 natives of Punjab (males).	1 Koni native of Orissa (male).	1 native of Cabul (male).	1 Sikh (male).
I. Lumbar Vertebra,	114·0	126·0	117·1	116·2	118·1	112·3	127·9	109·2	116·6
II. Lumbar Vertebra,	103·7	114·0	115·0	112·7	104·0	111·0	116·3	111·3	112·0
III. Lumbar Vertebra,	97·3	112·0	111·3	104·1	106·2	105·1	114·0	111·3	112·2
IV. Lumbar Vertebra,	97·0	115·2	101·0	86·5	94·0	101·1	108·0	105·7	104·0
V. Lumbar Vertebra,	87·5	86·7	93·6	86·9	78·0	93·0	92·0	92·3	93·6
Lumbo-Vertebral Index, . . .	99·7	110·5	107·6	101·0	100·0	104·3	111·5	105·9	107·5

The data which are given in this Table are of two meagre a kind for us to generalize upon. If it should so happen, however, that they are borne out by subsequent observations, the European will present a form-adaptation of the bodies of the lumbar vertebræ, more suited to the lumbar convexity than that exhibited by any of the other races which have come under our notice.

In bringing the first Section of this part of the memoir to a conclusion, I must express my indebtedness to my assistant, Dr. H. St. John Brooks, Demonstrator of Anatomy in the University of Dublin, for the ready help which he has given me in calculating the formidable rows of vertebral indices.

Section II.

The second section of this Memoir deals with the Lumbar Curve as a whole in man and the apes; and at the very outset of the inquiry we are met with a difficulty which has presented itself to every observer who has entered upon this subject. It is this: What standard are we to assume as typical? Is there any condition of vertebral curvature which may be considered normal? This is, indeed, a puzzling question, and we need not wonder, when we reflect upon the many forms which the spine may assume, that certain anatomists consider that there is no one condition which may be regarded as typical. In life[1] there is not a gesture, not a change of attitude, which is not accompanied by some alteration of spinal form. Our very respiratory movements, gentle and unconsciously performed as they are, meet a response in alterations of curvature of the vertebral column. And in the dead, when the spine is removed from the influence of the muscles—those contracting guy-ropes which help to keep it erect—when removed also from the influence of gravity, the changes which it undergoes must be considerable.

But even assuming that in life one of the many postures which may be taken by the human body is the normal one—be it the military attitude, or the easy unrestrained erect posture, it does not matter which—how is it possible for us to determine with accuracy the outline of the column formed by the bodies of the vertebræ? It is a task beyond the power of the anatomist. The only parts of the spine which approach the surface are the tips of the spinous processes, and the distances which remove these from the central points, or the anterior surfaces of the vertebral bodies, are not uniform. Not only are they different in different individuals, but even in dealing with persons of equal stature and development, and of corresponding sex, they cannot be relied upon. Further, in the movements of the spine, the excursions which are performed by the spinous processes are

[1] Henke remarks that the spinal form is variable according to taste and fashion.

much greater, and of a different nature from those of the bodies of the vertebræ.

Had it been possible to determine with precision the curvature of the spinal column in the living, there cannot be a doubt but that Parow[1] would have solved the question. His investigation into the physical conditions of the upright posture, and the normal curves of the spine, is a model of patient and laborious research; but while he has furnished us with many important and interesting facts, the representations which he gives of the curves of the column of vertebral bodies in the living can only be accepted as approximately correct.

The following is the manner in which Parow endeavoured to ascertain the curved outline presented by the anterior face of the vertebral column in the living individual. He tried to obtain the mean distance between the tips of the several spinous processes and the middle points of the front surface of the corresponding vertebral bodies by measurements carried out on a number of male and female columns. In certain of the dorsal vertebræ, on account of the strong slope of the spinous process, it was necessary to take three distinct measurements, viz. to the anterior face of the body of the corresponding vertebra; to the front of the vertebral body next in order below; and to the front of the column immediately opposite. In the cervical region such measurements would have been useless, not only on account of the deep position of the spinous processes, but also on account of their very variable degree of development. In the diagrams, therefore, which he gives, the cervical curve must be considered as being more or less fanciful. In the male spines the differences in the distances between the tips of the spinous processes and the front surface of the column were less than would have been expected. The most striking discrepancy, in the case of the movable vertebræ, was that exhibited by the vertebra prominens, and it amounted to 12 m.m. In the female spines the discrepancies were much greater, and the want of uniformity in different spines very marked.

[1] "Studien über die physikalischen Bedingungen der aufrechten Stellung und der normalen Krümmungen der Wirbelsäule." Archiv für pathologische Anat. und Phys. 1864.

But the great problem which presented itself to Parow was the determination of the position of the promontory—a point of the utmost importance in constructing a schema of the curvatures. The variable development of the sacrum renders it impossible to attain a definition of this point by measurements from the spinous process of the first sacral vertebra. In the two males measured by Parow the difference was so great as 27 m.m. He therefore had to devise other means of obtaining information upon this point, and not very successfully, for he says—" Ich habe indess in den dieser Abhandlung beigegebenen Zeichnungen den Versuch einer genauer Bestimmung der Lage des Promontoriums, resp: des Mittelpunktes der oberen Fläche des Kreuzbeins unterlassen, da ich mich überzeugt habe, dass dieselbe nicht die Wichtigkeit hat, die man ihr früher beilegen mochte."

Taking into account, then, the variations in form of the vertebræ, it will readily be understood that the difficulties in the way of obtaining the accurate curvatures of the spine in the living are insuperable. Parow carried out his investigations in this direction by means of an instrument which he terms the " Co-ordinatemesser," and unquestionably his results are of value, in so far as they bring out the transient changes of form which the vertebral column undergoes under the influence of slightly altered conditions. In fig. 3 (A and B) two of his outlines are reproduced. A represents what he considers to be the " Profile-projection of the fore surface of the vertebral column in the unrestrained upright attitude in the nearly empty condition of the alimentary canal." B is the same individual, with "a full condition of the alimentary canal." C and D are outlines of the fore surface of the spines of No. 1, male, and No. 3, female, in Plate ii. The latter are tracings of mesial sections of frozen isolated spines, in which, therefore, the different levels can be marked off with the greatest accuracy. It will be seen that in C and D, although they are drawn upon a larger scale than A and B, the distance between D 2 and D 6 (*i.e.* the second and sixth dorsal vertebræ) is virtually the same. This shows that, in so far as these points are concerned, Parow cannot have succeeded in placing them accurately upon his profile outline. And this discrepancy cannot be accounted for by a compression of the

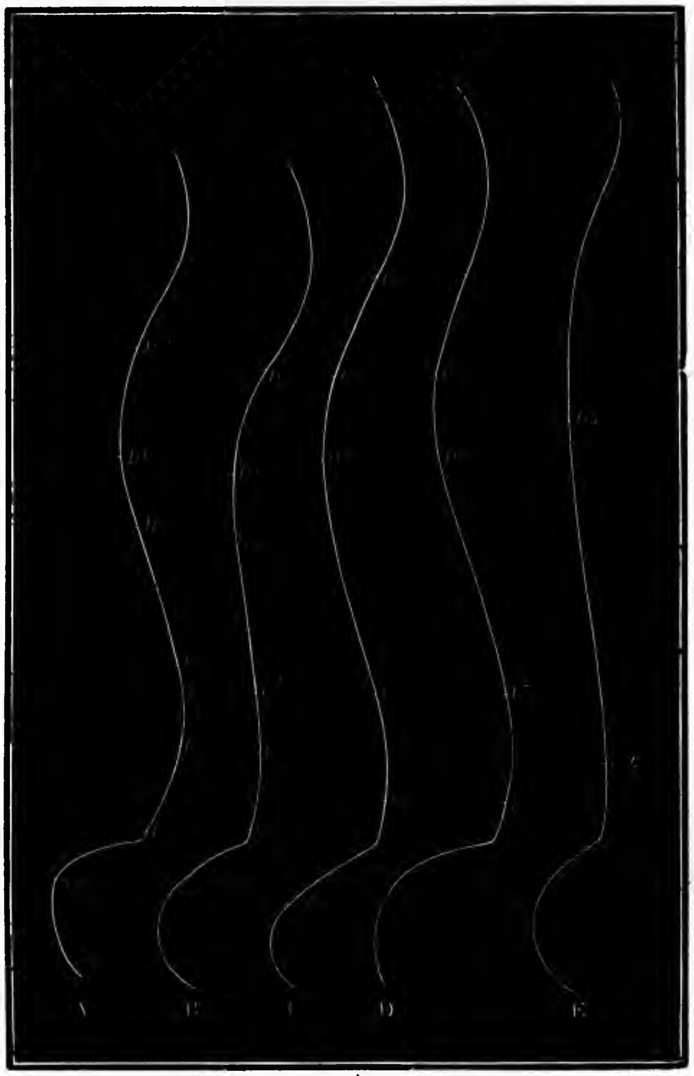


Fig. 3.

A and B (from Parow) :—

- D. 2. Upper border of the second Dorsal Vertebra.
- D. 6. Upper border of the sixth Dorsal Vertebra.
- D. 9. Middle of the ninth Dorsal Vertebra.
- L. $\frac{1}{2}$. Intervertebral disc between the first and second Lumbar Vertebræ.
- L. 3. Upper border of the third Lumbar Vertebra.
- L. 5. Lower border of the fifth Lumbar Vertebra.
- S. 3. Third Sacral Vertebra.

C Anterior curvature of No. 1, male, in Plate II. :—

D Anterior curvature of No. 3, female, in Plate II. :—

- D. 2. Middle point of the second Dorsal Vertebra.
- D. 6. Middle point of the sixth Dorsal Vertebra.
- D. 9. Middle point of the ninth Dorsal Vertebra.
- L. 3. Middle point of the third Lumbar Vertebra.
- L. 5. Middle point in the fifth Lumbar Vertebra.

E (from Parow) :—

- C. 5. Middle point of the fifth Cervical Vertebra.
- D. 7. Middle point of the seventh Dorsal Vertebra.
- L. 4. Middle point of the fourth Lumbar Vertebra.

Prom. Promontory.

four intervening discs by the superincumbent weight from which C and D have been relieved. This would never cause such a marked difference.

But even supposing that by the "Co-ordinatemesser"[1] we were able to obtain the precise curvature of the spine in the living, it would give a standard which could never be applied to comparative anatomical research. We are therefore obliged in this field of work to seek our standard of comparison in the dead.

Various methods have been employed for the purpose of determining the normal spinal curvature in the dead body. The brothers Wilhelm and Eduard Weber, in their classical treatise, "Ueber die Mechanik der menschlichen Gehwerkzeuge," published in 1836, were the first to treat the subject in a thoroughly reliable and scientific manner. Having removed the viscera and muscles from the trunk of a male subject, without injuring the ligaments of the spine and thorax, they embedded the preparation in plaster of Paris. When the latter was firm, the vertebral column and the block of enclosing plaster were divided in the mesial plane by means of a saw. By this method the various segments of the spine were held firm in the gypsum, and no displacement of these could take place through the escape of the fluid central parts of the intervertebral discs. The authors give a full-size representation of the cut surface of this spine, and I have taken the liberty of reproducing it in a reduced form in Plate ii. There cannot be a doubt that this method will give fairly accurate results, although it is more than possible that in the process of embedding the spine in the plaster a change in the norma lcurvature may be induced; and some authors consider that the strong dorsal concavity[2] which is exhibited in the Webers' representation is due to this cause.[3] Horner objects to this method as one calculated to give rise to inevitable errors from the manipulation

[1] Pansch, in his lectures to advanced students, remarks, with great truth, that in order to reproduce the spinal curvature of the living we would require to examine the individual first during life, and subsequently after death.—("Anatomische Vorlesungen.") 1884.

[2] The strong dorsal bend of this spine is certainly remarkable; but it appears to me that it is more reasonable to assume that it represents a pathological condition—perhaps, indeed, a condition resulting from emphysema of the lungs.

[3] Vrolik, "Tijdschr. voor de Natuurk. Welensch." T. iii. 1850.

and the weight of the plaster. Professor Humphry of Cambridge, in his work upon the "Human Skeleton," figures the mesial section of a spine which he had treated in the same way, and Dr. Pierre Bouland[1] likewise adopted this method in his investigations into the development of the spinal curves. But however suitable this plan may be for the adult spine, in which the curves are more or less consolidated, it must be very evident that it is altogether inappropriate for the flexible column of the new-born child.

Dr. Friedrich Horner[2] approached the question from a totally different point of view. In Weber's method no account whatever is taken of the influence exerted upon the spine by the force of gravity, or the elasticity of the muscles, and Horner endeavours to rectify this. He does not attempt to arrive at an exact result for any one spine, but endeavours to construct an ideal or typical curve of the fore surface of the vertebral column from results obtained by the examination of three specimens. In two of these he left the muscles intact, whilst in the third the ligaments alone were retained. The vertebral columns thus prepared were divided in the mesial plane, and one-half of each placed upon a board. The sacrum was then firmly nailed in position, and the two end-points of the fore surface of the section connected by a straight line. The different regions were investigated separately, and by bending each part backwards and forwards its maximum power of backward and forward movement, as well as its middle position, were determined. Horner further endeavoured to compensate for the absent influence of gravity by pressure upon a given point; and in order to determine this point he employed the law which had been propounded by H. Meyer, viz. that the upright position "is maintained essentially only by conditions which lie in the skeleton and its ligamentous apparatus, and by which only the smallest degree of muscular activity is called into play." Meyer places the point of gravity of the trunk with the head and arms opposite the ninth dorsal vertebra, whilst he considers that the vertical line of gravity passes

[1] "Rech. Anat. sur les courbures du Rachis chez l'homme et chez animaux."—Journ. de Anat. Paris, 1872.

[2] "Ueber die normale Krümmung der Wirbelsaüle." Müller's Archiv. 1854.

downwards so as to intersect the third sacral vertebra. In Horner's representation, therefore, the anterior face of the ninth dorsal vertebra is brought back until it touches the vertical line of gravity. It is by no means to be regarded as certain that the premises assumed by Horner are correct. The line of gravity is a point of extreme difficulty to determine, and Parow, with a full acquaintance of the work of Meyer and Horner, considers that it is not placed so far back, but cuts the promontory. It is in this neighbourhood also that Weber places the line of gravity.

In a postscript to Horner's Paper, H. Meyer makes a few slight alterations in Horner's schema, and likewise gives a formula by means of which he considers that the curves of an ideal and typical vertebral column may be constructed.

Pansch,[1] in his instructive and interesting lectures to advanced students, remarks: "A single determined form (of the vertebral column), which we might designate a normal one, does not exist. . . . But with Meyer we regard that form, which the powerful and stiff upright body shows, as a typical one. This stiff or 'military' position . . . offers the peculiarity that the two ends lie perpendicularly, the one over the other." If the Horner-Meyer schema represents the military position, then it is a forced attitude, and one which is very largely produced by the influence of muscular contraction. Krause[2] indeed considers that this posture is only fit for a pregnant woman.

Fr. Merkel[3] objects to the preceding methods of investigating the curves of the vertebral column, inasmuch as they only give us information regarding the vertebral bodies, and the intervening cartilaginous discs. We obtain, in other words, the collective curvature without perceiving the part which the several vertebræ play in its construction, because the neural arches, and more especially the articular processes, are concealed from view: and he holds that it is only by a study of the details of the architecture of the separate vertebræ that we are able to tell whether or not a normal curvature exists. His Paper on this subject is an exceedingly ingenious one. By the application of geometrical laws he has been able to construct

[1] "Anatomische Vorlesungen." 1884.

[2] "Handbuch der menschlichen Anatomie," p. 81.

[3] "Ueber den Bau der Lendenwirbelsaüle." Archiv für Anat. und Phys. (His und Braune). 1877.

drawings of the lumbar region of twelve different individuals from the macerated and separate vertebræ. He endeavours, therefore, to rebuild the lumbar section of the column from a study of the manner in which its separate elements are related to each other.

The following is the plan upon which he proceeds, and it would be well to use his own words. He says: "It is necessary to call to our aid a mediate method of investigation. As such the excellent method of geometrical drawing of Lucæ has proved itself very valuable to me. With its help it is possible to sketch a profile outline of the vertebræ which comprehends in one plane all the essential points of the bodies as well as the arches, and thus unites the advantages of a median section with those of a view of the arches. With a little practice we obtain such accurate drawings, that several independent sketches prove, in every respect, congruous.

"In the production of the sketches I proceeded to place the vertebra, with the help of the '*Fadenkreux-Diopter*,' in accurate profile; the separate lumbar vertebræ of the macerated vertebral column, which I used exclusively, were grasped by a vice, and shifted until the upper and lower posterior angles of the body, as well as the highest and lowest point of the two articular surfaces of both segments of the body, coincided. The sketch was then drawn, and the superior articular surface was represented in the interval which exists upon the averted side of the articular process. This could also be very accurately done by using the contour of the processus obliquus, and by means of compass measurement. Finally, in the drawing, likewise by the aid of the compass, the middle points of the upper and lower articular surfaces were determined."

Having thus obtained accurate profile drawings of the five lumbar vertebræ, Merkel proceeds to place them in position, and thus re-construct the lumbar column. He assumes, in the first place, that in the attitude of anatomical rest the central points of the opposing surfaces of the articular processes must coincide with each other, and having placed them accurately in this relation to each other, he shifts the bodies until the upper and lower angles of their fore surfaces fall into a uniform curve (*i.e.* a curve without angular notches). He asserts that it is only possible to form, with the

front faces of the bodies, one quality of curve. If we attempt to construct a curve with a larger radius, then the central articular points refuse to coincide: if, on the other hand, we strive to form a curve with a smaller radius, the vertebræ project over one another. He figures the lumbar regions of twelve individuals which he has reconstructed in this way, and he considers the curve in each case, "die einsig mögliche anatomische Normalkrümmung der Bauchwirbelsaüle."

He brings forward additional evidence to show that at least the second and fourth lumbar vertebræ are correctly placed. If the lines joining the central points of the superior and inferior articular processes of these vertebræ be prolonged, they cut each other at an angle of 166°—and this angle is constant throughout the whole series of columns he examined. The position of the third vertebra varies with the height of the individual, but he also endeavours to deduce a law by which its position may be determined. It is only in the case of the first and fifth lumbar vertebræ that he finds himself placed in a dilemma. In the case of the former, he seeks help from the Horner-Meyer schema: as for the latter, its place was fixed, more or less, by the eye. With regard to these vertebræ, he says: "Ich betone nochmals, dass diese Stellung nur eine provisorische ist, und dass sich volle erst Sicherheit über diesen Wirbel, ebenso wie über den ersten Lendenwirbel ergeben kann, wenn eine Construction der ganzen Wirbelsaüle vorliegt."

Were it possible to accept without question this highly ingenious method of rebuilding the lumbar column, the advantages proceeding from it would be very great; as Professor Merkel claims, it would enable us to appreciate the relation which the several parts of a vertebra bear individually to the curve, and it would supply us with a ready means of estimating the degree of lumbar curvature in the macerated column. It might, therefore, be employed with great advantage in the determination of the curves of those races of which the fresh spines cannot be obtained. There are, however, decided difficulties in the way of its application. The axiom which is advanced, viz., that the position of anatomical rest is that in which the central points of the opposing articular surfaces coincide, might readily be disputed; and upon this depends the whole theory. But, even granting the accuracy of the method, it has this disadvantage, that by its means we

are only able to fix with certainty the position of two, or at most three, of the vertebræ. Whilst we recognize, therefore, the high importance of this Paper, and note the many original and new points which it discloses, we are satisfied that the curves of the spine can only be properly studied in the fresh vertebral column.

The investigations which I have made in connexion with the lumbar curve in man were chiefly carried out with the view of obtaining a standard with which I might compare the corresponding curve in the apes. It is true that I might have been satisfied for this purpose with the representations of Weber,[1] Braune,[2] and Pirogoff;[3] but as it was important to arrive at some conception as to the range of deviation in the lumbar curve in different individuals, I felt that it was necessary to examine a number of spines. The investigation has not been devoid of interesting results.

In every case the spine was frozen, and divided in the mesial plane with the saw; a piece of fine tracing-paper was then frozen to the surface of the section after it had been cleansed with a stream of water, and an accurate tracing taken of the bodies of the vertebræ. There cannot be a doubt that where it is possible it is advantageous to freeze the entire body, and thus obtain a picture of the spine before it has been removed from its surrounding soft parts; but in a country such as Ireland, where a snowstorm is a rarity, and a continuous hard frost seldom occurs, it is out of the question to attempt the freezing of a number of entire subjects.

The spines, therefore, which I have examined, were removed from the body, but in every case the surrounding muscles and ligaments were left untouched. I have satisfied myself that the changes of curvature which Parow[4] asserts occur when the spine is isolated do not take place when the subject is maintained in the horizontal position. It is a totally different matter, however, when we are dealing with a subject which is artificially maintained in an erect attitude. In this position it would be strange if the

[1] *Antea*, p. 3.
[2] "Atlas of Topographical Anatomy."
[3] "Anatome topogr. sectionibus per corpus hum. congelatum triplici direct. ductis illustr. Petropol." 1859.
[4] "Archiv. für pathologische Anat. und Phys." 1864.

changes described by Parow, on removal of the viscera and the thoracic wall, did not take place. It must be evident to everyone, that if in a dead body, held in an upright posture, the abdominal viscera be removed—if in this way the support to the under surface of the diaphragm be abstracted—the upper part of the dorsal region of the column will sink downwards, and at the same time advance forwards: likewise it must be equally clear that the evisceration of the thorax will cause the spine to partially regain its original form. Further, if the ribs and sternum be taken away, and the dorsal region be thus relieved of their weight, it is reasonable to suppose that an increased flattening of the dorsal curve will follow. But all these changes are due to the force of gravity, and exert no influence upon the spine when it is placed in the horizontal position.

But Parow holds that the thoracic walls constitute an integral part of the spine, and that the rings formed by the costal arches and the sternum keep the dorsal region of the vertebral column in a state of tension. He asserts that as soon as the integrity of the thoracic parietes is interfered with, the dorsal arch relaxes, becomes flatter, and forms a curve with a distinctly greater radius. The lumbar curve also completely disappears.

To test the accuracy of the above statement I performed the following experiments:—Having chosen a powerfully-built male subject, the lower limbs were amputated at the junction of the upper and middle thirds of the thighs, and the head removed by disarticulation at the occipito-atlantoid joint. The upper limbs, with the shoulder-girdles, were also taken away. The trunk, thus bereft of the greater part of the lower limbs, the upper limbs, and the head, was placed upon its back in a box about twelve inches longer, and about eight inches broader than itself, and plaster of Paris was poured in so as to fill up the vacant space. In the cervical region this was allowed to rise until it reached the level of the anterior surface of the vertebral bodies; in the dorsal region it was allowed to come forward upon the chest wall to a point a little beyond the nipples, and in the abdominal region until it reached the anterior surface of the front wall. The trunk was therefore immovably fixed in a block of plaster, with its anterior surface alone exposed. Great care was taken to regulate the amount of plaster behind the neck, so that it might corre-

spond to the depth of the head behind the atlas. The viscera of both abdominal and thoracic cavities were then removed by opening the anterior wall of the former: the thoracic wall was in this way preserved intact. During the process of evisceration the spine was supported by the hand of an assistant, so as to prevent any dislocation from the plaster bed. The subject, and also the spine, remained, however, perfectly firm, and did not move in the least degree. After cleaning the front of the vertebral bodies, it was noticed that the first and second lumbar vertebræ had been the seat of caries, and although completely cured, there was yet a distinct depression opposite this portion of the column. The curve was taken by a strip of lead moulded upon the front face of the vertebral bodies, and the tracing thus obtained is represented in fig. 4 (No. 2 A.) It may be regarded as giving the curvature of this spine with the body cavities entire, and occupied by the viscera.

The next step in the experiment consisted in breaking down the plaster block, with the exception of that portion which supported the neck, which was retained throughout the entire experiment. When this was effected the trunk was again placed in the box and again surrounded with plaster, as before, and the curve taken with the strip of lead. The tracing thus obtained is seen in fig. 4 (No. 2 B.) It gives therefore the outline of the spine in the horizontal position after evisceration.

Before breaking down the plaster block a second time the breast-bone and the costal cartilages were removed, and having again embedded the body in the gypsum, the tracing represented in fig. 4 (No. 2 C) was taken. This gives the curvature after the integrity of the thoracic wall has been interfered with.

The last step consisted in freezing the isolated spine, and dividing it in the mesial plane. A tracing (in a reduced form) of this section is given in fig. 4 (No. 2 E.)

The pathological condition of this vertebral column rendered it necessary that the same experiment should be repeated upon a healthy subject; and a tall powerful male was therefore chosen, and subjected to precisely the same process. The only difference in this case was, that the lower limbs were not amputated. In the second embedding the plaster block

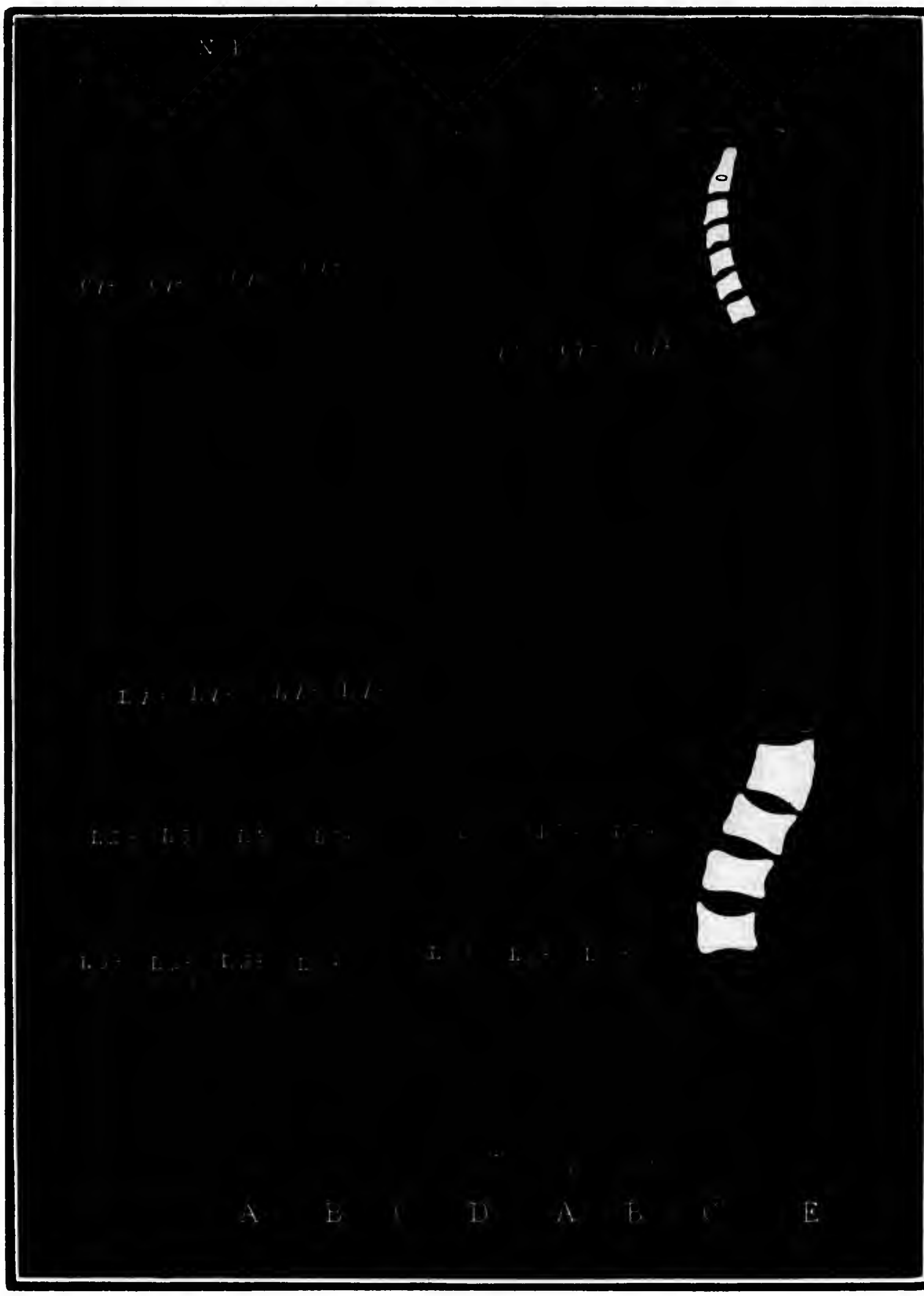


Fig. 4.

supporting the head broke, and thus some differences resulted in the two last tracings of this portion of the column. The curves obtained are represented in fig. 4 (No. 1 A, B, C.) An additional outline is given, viz. D; this was the tracing obtained after all the intercostal muscles, &c., were divided right back to the erector spinæ, and in taking this the trunk was not embedded in the plaster, but was merely stretched horizontally on a table.

It will be seen that the tracings of each spine are very nearly alike, and that the deviations are so slight that they are evidently the result of the method of registering the curves, viz. by a strip of lead. I believe, therefore, that in the horizontal position of the body the removal of the viscera from the abdominal and thoracic cavities does not appreciably affect the curves of the vertebral column; and further, I dispute Parow's proposition that the dorsal region is held in tension by the costo-sternal rings, and that, when the body is in the recumbent attitude, any interference with the integrity of the thoracic walls leads to an alteration of the vertebral curves. No one will deny that in the upright position they exercise an influence by reason of their weight, and that in the movement of inspiration they produce a slight straightening of the dorsal part of the column.

Parow figures the outline of the front surface of a spine which has been isolated, and placed in the horizontal position; and this I have taken the liberty of reproducing in fig. 3 E (p. 40). It is remarkable for the extreme flatness of its curves, more especially in the lumbar region, and at the promontory. Two of the spines which are represented in Plate II. (viz. No. 4 and No. 5, males) are also peculiar on account of their straightness; but this cannot be due to their having been isolated before they were frozen, because the other members of the series were treated in identically the same way. As a contrast to Parow's outline, I place in fig. 3 the outlines of the anterior faces of No. 1, male, and No. 3, female, in Plate II. They are indicated by the letters C and D.

I am convinced, therefore, that isolation of the spine in the recumbent posture does not produce the changes described by Parow as occurring when the trunk is maintained erect. In the series of spines figured in

Plate ii. I have introduced reduced tracings of the spines in Braune's representations of the mesial sections of a male and female (No. 2, male, and No. 5, female). Certain of my own tracings closely correspond with these: thus No. 1, male, only differs from Braune's male in the cervical region, and this is evidently due to the position of the head in the latter; again, if No. 3, female, be contrasted with Braune's female, a similar correspondence will be noted.

Braune[1] is unquestionably right when he states that no advantage can be obtained by freezing the body in the upright position. "It is evident," he says, "that it is impracticable to keep a body so balanced, and in such equilibrium, as the muscles are capable of doing during life. The trunk always hangs over to one side to such an extent that the spine partly loses its original curvature, and takes a semi-flexure. It is, therefore, not to be wondered at that the figure which Pirogoff (a. a. O. Tab. 12) gives, taken from a subject frozen in the upright position, exhibits flatter curves, having flatter arcs, than it would have had if the drawings had been taken from one frozen in the horizontal position."

Of course it is impossible to estimate accurately the amount of alteration which the spinal form undergoes in life in the change from an erect to a recumbent attitude. We can only judge of this by measuring the spine in the two attitudes, and the results obtained in this way would certainly seem to indicate that the change of curvature is not so marked as is generally supposed. At all events we are justified in assuming that in a dead subject the spinal form in the recumbent posture approaches more nearly to what it was during life than when the body is artificially maintained in the erect attitude.

In order to obtain an approximate idea as to the amount of the difference in the spinal curvature in the two attitudes, I carefully measured six young, well-formed individuals, with an average age of twenty-eight, and an average height of 5 ft. $6\frac{1}{2}$ in., first in the upright posture, and afterwards in the recumbent position. The average increase in height in the latter attitude was only three-eighths of an inch, no doubt produced by a

[1] "Atlas of Topographical Anatomy," p. 3.

flattening of the dorsal and lumbar curves.[1] A corresponding increase in height can be detected, as is well known, after a night's rest. Hyrtl[2] sta... that in measurements instituted upon himself he found his height to ... "5 schuh 8 zoll" after seven hours' rest, and only "5 schuh 7 zoll 3 linien" before retiring to bed. In eight individuals (four males and four females), with an average age of twenty-seven, and an average height of 5 ft. 4 in., I found the morning increase of height to be an average of $\frac{7}{10}$ of an inch. This increase depends not so much upon an alteration of the intervertebral discs as upon a change in the curves of the spine, which, during the period of rest, has been relieved from its burden.

It therefore appears to me that the attitude in which the spine should be examined in the dead subject is the horizontal recumbent one, and the spinal form which is then exhibited may fairly be taken as the standard of comparison when we are pushing our inquiries into the curves of the vertebral column in the lower animals.

Differences in Spinal Form in different Individuals.—It is not necessary to examine the spine itself to observe the marked differences in spinal form exhibited by different individuals. The stooping gait of those engaged in sedentary pursuits, the erect bearing of the soldier, and the easy and graceful carriage of the athlete, are all types familiar to the eye. Still these individual differences are brought very prominently before us when we examine a series of spines such as are exhibited in Plate II. The influences at work in producing these differences are clearly those of—(1) age; (2) sex; and (3) occupation.

It is recognized by all anatomists that the spinal form presents different appearances at different periods of adult life. According to Sappey,[3] the curvatures of the vertebral column remain stationary only for the short period of five years, viz., between the ages of twenty-five and thirty.

[1] Horner believes the difference to be much greater. He took two points, viz. one opposite the middle of the third sacral vertebra, and the other at the tip of the spinous process of the eighth dorsal vertebra, and measured the distance between them, first in standing, and then with the subject lying upon his side, and found a difference of 30 mm.

[2] "Lehrbuch der Anatomie."

[3] "Traité d'Anatomie Descriptive."

"The curvature then enters upon a new period of increase, and the stature re‸ns to decrease." But it is only in advanced age that the curves ‸ome greatly exaggerated, and this exaggeration does not affect the lumbar so much as the dorsal region. In Plate II., No. 1 (male) represents the spine of a man of about fifty-five years; Nos. 3 and 6 (males) are the spines of men of middle age (from forty to fifty years); whilst Nos. 4 and 5 (males) are the spines of, comparatively speaking, young men. Confining our examination to the lumbar region, it will be observed that the most pronounced curves are to be seen in two out of the three oldest spines, viz., Nos. 1 and 6. By referring to Table R. (p. 27) it will be observed that the index of the lumbar curve in these is 10·3 in the former and 12·9 in the latter. No. 3 certainly offers a marked exception, the index being only 6·9. But there are two other peculiarities in connection with this spine, viz. it presents a most unusual abruptness in the transition from the cervical to the dorsal curve, and the average index of the bodies of the lumbar vertebræ is above the standard (101·1).

In the series of female spines represented in Plate II., No. 1 is from a woman thirty years old; No. 3, from a subject thirty-five years old; whilst Nos. 2 and 4 were both about fifty years old. In general curvature the two young spines are undoubtedly the most pronounced, and although No. 4 presents the highest index of lumbar curve (12), No. 2 possesses the lowest (6). The straightness of this latter spine is indeed very striking.

The influence of age, therefore, does not affect all spines, nor all regions of the spine alike. Although there was no decrepitude, and, indeed, no apparent wasting in the subjects from which Nos. 3 (male) and 2 (female) were taken, it is possible that they may have remained in a recumbent position for a longer period than the others before death.

That the life-occupation of an individual exerts an influence upon spinal form must be manifest to all, and doubtless many of the peculiarities observed in the series of spines figured in Plate II. proceed from this cause. Unfortunately, however, with the exception of No. 5 (male) (who, I believe, was a sailor), I have no information as to the pursuits followed by the individuals from whom the spines were taken. Sappey[1] believes that in

[1] *Antea*, p. 52.

the agricultural labourer the upper part of the lumbar region becomes more strongly curved; whilst in the porter it is the lower part of the cervical region that is affected. "Telle est aussi," observes Charpy,[1] "la cambure des gens qui habitent la montagne ou des pays accidentés et marchant sur des pentes, opposée à la rectitude de ceux qui vivent en plaine." All such forms are the result of muscular influence.

The influence of sex upon the general lumbar curve has already been discussed, and it has been noted that in the female it is, as a rule, more pronounced than in the male. Other special sexual distinctions will be alluded to when we come to analyse the character of the lumbar curve.

Racial Differences.—It cannot be disputed that in different races of mankind there are different varieties of spinal form. The very fact of such marked discrepancies being found between the indices of the bodies of the lumbar vertebræ points to this, although it cannot be regarded as a proof of it. As we have already observed, these discrepancies can only be looked upon as indicating a greater or less degree of flexibility—a greater or less degree of power of changing the spinal form under the influence of the muscles, and not as specially indicating a greater or less degree of lumbar curvexity.

Travellers tell us of the erect and graceful bearing of many of the natives of Africa, and speak of the marked hollow which may be seen posteriorly in the region of the loins. The same has been said also of the natives of Australia; but beyond this we have little information, and what knowledge we do possess is far from being of a definite character.

Pruner-Bey,[2] in his memoir upon the Negro race, says: "Les trois courbures de la colonne vertébrale ne sont jamais aussi prononcées que chez le Touranien et chez l'Aryen." Topinard,[3] in his great work upon Anthropology, makes the remark that the *l'ensellure dorso-lombo-sacrée* "est plus prononcée dans les races brunes méridionales de l'Europe par comparaison avec les races blondes et cela dans les deux sexes."

[1] "De la courbure lombaire."—Journ. de l'Anatomie et Physiologie. (Paris.) 1885.

[2] "Mémoire sur les Nègres," lu à la Société d'Anthropologie de Paris. 1861.

[3] "Eléments d'Anthropologie générale." 1885.

Duchenne[1] gives some very interesting facts regarding the difference in spinal form in the Iberian and Anglo-Saxon races. He states that three very different degrees of lumbo-sacral curve exists, and to the most pronounced of these he gives the name of "*Ensellure lombo-sacrée physiologique.*" This condition he considers is characteristic of Spanish women, and more especially of the Andalusians. It necessitates curves of compensation in the other regions, and, when these are not exaggerated, much grace is given to the contour of the body: further, it is associated with small hands and feet, elegant pose, and finely-modelled neck. The same characters may be observed in the women of Boulogne-sur-Mer, which he believes to be due to their supposed Spanish origin. In the village of Andresselles, with inhabitants of Anglo-Saxon blood, he has noticed an altogether different type. In these the vertebral column is very straight, and the lumbar curve but slightly marked. Associated with this variety of spinal form there was noticed a stiff body with an angular contour, large hands and feet, and a neck wanting in grace.

M. le Dr. Guerlain,[2] a physician in Boulogne-sur-Mer, disputes the accuracy of the views advanced by Duchenne. He holds that the cause of the *Ensellure lombo-sacrée* is mechanical, and due to the hilly nature of the district and to the heavy burdens which the inhabitants carry. Lagneau,[3] in his Paper upon the Anthropology of France, gives his adherence to the opinions of Duchenne.

I have recently had an opportunity of examining a number of Hottentot-Bushmen, who were exhibited in Dublin by M. Farini. The troupe consisted of two adult males, a boy of six years old, and a girl of twelve years old. In all, the depression in the lumbo-sacral region of the back was very marked, and rendered all the more striking by the great accumulation of gluteal adipose tissue. The lower part of the abdomen was also very protuberant. With the help of an assistant I obtained accurate

[1] "Physiologie des Mouvements." 1867.

[2] "Sur l'ensellure lombo-sacrée des femmes de Boulogne."—Bulletin de la Soc. d'Anthrop. de Paris. 1867.

[3] "Bulletin de la Soc. d'Anthrop. de Paris." 1867.

tracings of the spinal form, in so far as this can be done in the living. Each individual was instructed to stand in an easy upright attitude, and being steadied in this posture, by the assistant placing one hand on the head and the other on the shoulder, I proceeded to mould along the middle line of the back a strip of ductile lead. Owing to the great projection of the buttocks, it was very difficult to obtain an accurate mould of the coccyx, but above this point the form could be taken with the greatest precision. Fig. 5 A, B, C, D (p. 58), gives, on a reduced scale, the tracings which were obtained in this way. The *Ensellure lombo-sacrée* is very pronounced, and it may be measured, as Topinard suggests, by drawing a line from the most prominent point of the dorsal region to the most prominent point of the sacrum, and then noting the depth between this line and the point of maximum curvature. Further, if we take the standard length of the spine as 100, we can formulate an index for the *ensellure lombo-sacrée.* The following Table gives the results of the measurements:—

FOUR HOTTENTOT-BUSHMEN.

Outline in Fig. 5	A.	B.	C.	D.	
Name and Sex,	N'Arkar, ♂	N'Icy, ♀	N'Fun n'fon, ♂	N'Con n'qui, ♂	Average.
Age,	6	12	25	42	
Length of Spine,	384 mm.	427 mm.	462 mm.	559 mm.	458 mm.
Maximum depth of *Ensellure lombo-sacrée,*	24 mm.	34 mm.	36 mm.	37 mm.	32·7 mm.
Index of *Ensellure,*	6·6	8·0	7·7	6·6	7·2

The female, although a girl of only twelve, presented the most marked curve, and this could be noted by the eye. In all there was observed a great suppleness in the region of the loins. It is right to mention, that although the age of N'fun n'fon was given as twenty-five years, it is very doubtful if he was more than twenty. He certainly had not the appearance of having arrived at full maturity.

In order to fully appreciate the above figures referring to the *ensellure* in the Hottentots, I measured in the same way two young adult and finely-built Irishmen, and the tracings which I obtained are represented in Fig. 5 (E and F.) The one was a labourer (E), and his spinal form was taken while he was standing in an easy, unrestrained attitude; the other was a tall, athletic man, with a very erect bearing (F), and in his case the tracing was obtained while he stood in an attitude with the shoulders drawn well back, and the loins projected forwards. The following Table gives the results:—

TWO IRISHMEN.

	E.	F.	Average.
Length of Spine, ...	682 mm.	741 mm.	711·5 mm.
Maximum depth of *Ensellure lombo-sacrée*, ...	21·5 mm.	41 mm.	31·2 mm.
Index of *Ensellure*, ...	3·2	5·5	4·3

The difference between the *Ensellure lombo-sacrée* in the Hottentot-Bushmen and the two Irishmen is therefore very striking, and bespeaks in the former not only a more pronounced lumbar curve, but, perhaps, also a greater obliquity of the pelvis. The index exhibited by F falls considerably short of the average Hottentot index, notwithstanding the fact that it must be regarded as showing a somewhat exaggerated *Ensellure*.

There is, however, still another point of difference, viz., that whereas the point of greatest depth of the *Ensellure* in the two Irishmen is opposite the second lumbar vertebra; in the Hottentot-Bushmen it is placed much lower down, and is opposite the last lumbar vertebra.

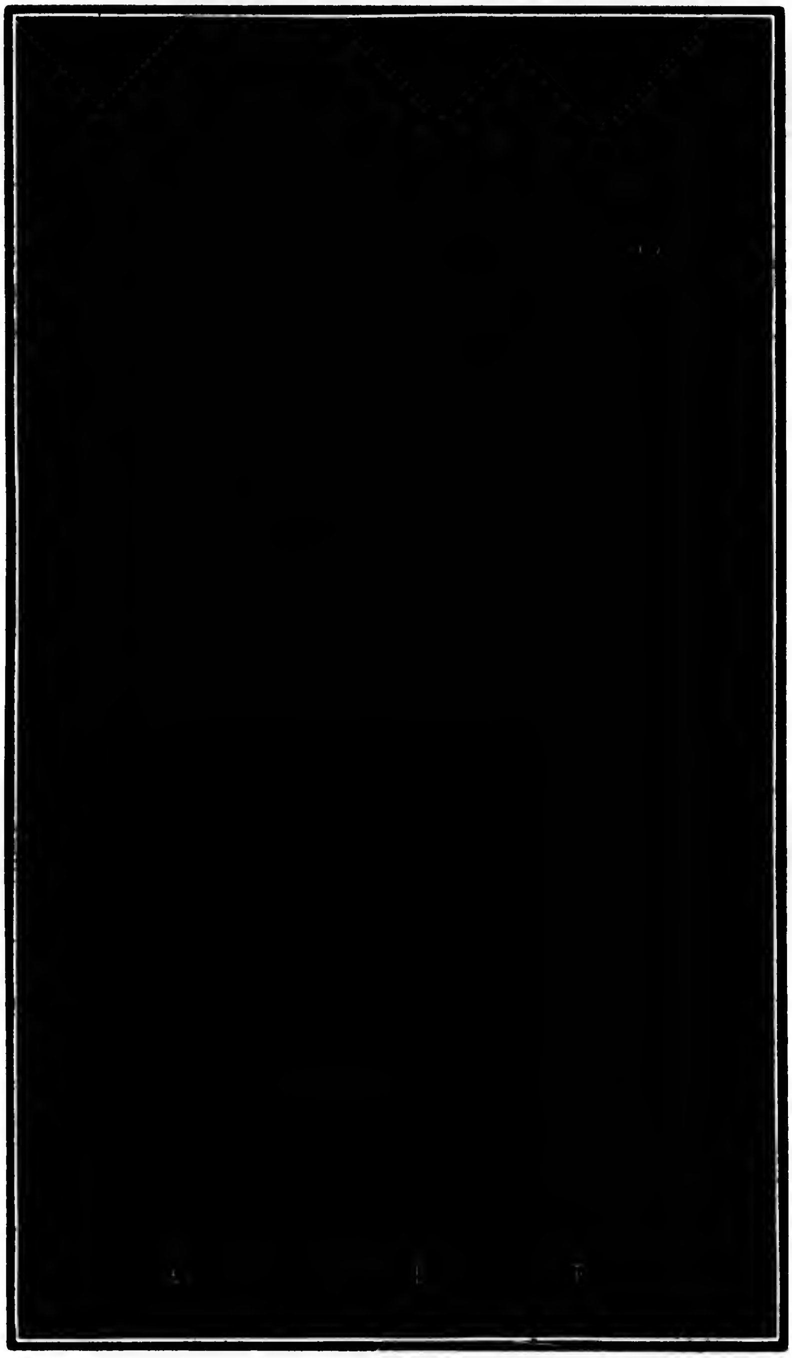


Fig. 5.

Tracings of the posterior aspect of the spine in four Hottentot-Bushmen, viz., A, B, C, and D; and two Irishmen, E and F; taken by means of a strip of lead.

Points of Inflexion of the Vertebral Column.—As mathematical problems are involved in the determination of this question, I sought the aid of the Rev. Dr. Haughton, F. R. S., whose previous well-known work in the field of animal mechanics rendered his help peculiarly valuable. With his usual kindness, he instructed me in the methods which he considered would lead to the most accurate results.

The putting of these methods into practice constituted a very laborious undertaking, and it was executed for me by my friend and former pupil, Dr. Vaughan. I cannot sufficiently express my indebtedness to him for the great assistance which he has given me, and for the many valuable suggestions which, from time to time, I received from him.

Hitherto the vertebral curvature has, as a rule, been regarded as that formed by the outline of the anterior face of the spinal column, and the mean curve has been constructed by drawing a line in relation to the upper and lower anterior angles of the vertebral bodies. This is manifestly an erroneous conception of the spinal curvature. That part of the column which is composed of the true vertebræ constitutes a cone-shaped structure, with a curved outline and a curved axis: its true curvature is that of the axis.[1] But further, the upper and lower angles of the fore surface of the vertebral bodies are very irregular in their degree of projection. Subsidiary curves are thus introduced which throw certain of the vertebræ out of the mean curve altogether. If we endeavour to determine the points of inflexion of the three curves by this method, the results obtained in different spines are by no means uniform. Thus the cervico-dorsal change takes place within a range which varies from the upper border of the third dorsal vertebra to the lower border of the fourth dorsal vertebra: whilst the lumbo-dorsal inflexion may occur at any point between the upper border of the first lumbar vertebra and the lower border of the third lumbar vertebra.

The true vertebral curvature is the *axial curvature;* and this may be determined by finding the central points of the bodies of the vertebræ and

[1] It is right to state that this point was first suggested by Dr. Vaughan in the course of a conversation which I had with him on the subject, and afterwards, in submitting the question to Dr. Haughton, he entirely corroborated the view.

of their interposed cartilaginous discs, and drawing the mean curve through these. When this is done, the change from one curve to the other will always be found at the same point; and it is of great interest to note that the degree of curvature of a spine appears to have no effect in producing variation in the points of inflexion: in a strongly-curved spine, as well as in a feebly-curved spine, the transition from one curve to the other is always at precisely the same spot.

In male and female spines alike (viz., Nos. 1, 3, 4, 5, and 6, males, and Nos. 1, 2, 3, 4, females, Plate II.), the cervico-dorsal point of inflexion is situated in the disc between the 2nd and 3rd dorsal vertebræ. The cervical curve is therefore formed by the upper nine vertebræ.

The lumbo-dorsal point of inflexion is apparently different in the two sexes, and therefore, in so far as my tracings show, it constitutes a sexual distinction. In male spines, Nos. 1, 3, 4, 5, and 6, the change of curvature takes place in the disc, between the 12th dorsal and the 1st lumbar vertebræ. In the female spines, Nos. 1, 2, 3, and 4, it is placed a little higher, viz., in the centre of the body of the 12th dorsal vertebra.

The most diverse views are entertained as to the boundary line between the dorsal and lumbar curves, and this may in all probability be accounted for by the fact, that in the determination of this point the mean curvature of the anterior face of the column has been taken as the standard. Aeby, in his work entitled "Der Bau des menschlichen Körpers," holds that the dorsal curve stretches from the middle of the cervical region to the middle of the lumbar region; and, in his memoir upon "Die Altersverschiedenheiten der menschlichen Wirbelsaüle," he asserts that a point between the second and third lumbar vertebræ constitutes the natural boundary between the concave dorsal and the convex lumbar regions. Humphry[1] and Barwell[2] state that the lumbar curve begins in the middle of the last dorsal vertebra and ends at the lower edge of the last lumbar. This closely corresponds with what we have found in the female spines. Gegenbaur, in his "Lehrbuch der Anatomie des Menschen,"

[1] "The Human Skeleton."

[2] "Lateral Curvature of the Spine." 1870.

inclines somewhat to the view enunciated by Aeby, and considers that the last cervical and the first lumbar vertebræ enter the dorsal concavity. In the most recent Paper upon the lumbar curve, viz., in the interesting article by Dr. Adrien Charpy[1] this point is discussed with great clearness. He says, "The lumbar curve of the adult has its convexity directed to the front. It comprehends, as a general rule, seven vertebræ, that is to say, the last two dorsal and the five lumbar; more rarely the lumbar alone, or the last nine vertebræ. It is not surprising to see the last two dorsal vertebræ enter into the movement of the lumbar column. Cruveilhier has remarked that, on account of the shortness of their spinous and transverse processes, and also on account of the extreme freedom of the floating ribs, they are in a condition very favourable for mobility; and Broca, who calls them the *false dorsal*, and attaches them to the lumbar system, observes that in quadrupeds they are really isolated from the true dorsal system; the movement of the posterior section of the column upon the anterior taking place exclusively between the last dorsal vertebra with a fixed rib, and the first of the false dorsal vertebræ."

Summit of the Lumbar Curve.—This is placed at a different point in the two sexes. In the case of the curve which is formed by the line drawn through the central points of the vertebral bodies and intervertebral discs, the point of maximum curvature corresponds in the male to the central point of the fourth lumbar vertebra; in the female it is placed at a higher level, and corresponds with the central point of the cartilaginous disc between the third and fourth lumbar vertebræ. No. 1 (female), Pl. II., is an exception to this rule. In this spine the first sacral vertebra is included in the lumbar curve, and the point of maximum curvature is the central point of the cartilaginous disc between the fourth and fifth lumbar vertebræ.

There is also a sexual difference in the point at which the summit of the curve formed by the anterior outline of the lumbar region is situated. In the males the most projecting point is the upper border of the fourth lumbar vertebra; in the females it is the lower border of the third lumbar vertebra.

[1] "De la courbure lombaire."—Journ. de l'Anatomie et Physiologie. Paris, 1885.

In his Text-book upon Human Anatomy, Gegenbaur states that the summit of the curve is to be found in the fourth lumbar vertebra; Henle[1] and Horner[2], on the other hand, place it on the disc between the fourth and fifth lumbar vertebra; and Henle refers to a drawing representing a mesial section of the vertebral column. In this, however, the summit of the curve is the same as in the male spines figured in Pl. II., viz., the upper border of the fourth lumbar vertebra. Dr. Adrien Charpy, in his article quoted above, considers that the point of maximum curvature varies in position, but that it is always situated in relation to a cartilaginous disc. He believes that, as a rule, it "is upon the disc which unites the third to the fourth lumbar vertebra"; but he has frequently found it between the fourth and fifth; more rarely between the second and third. In the spines which I have figured in Pl. II., we see every degree of curvature, and only one variation in the point of maximum curvature (No. 1, female), and this is entirely due to the peculiar form of the sacrum.

Composition of the Lumbar Curve.—If we analyse the axial curve of the lumbar region we find that it is not composed of the segment of *one* circle, but of segments of *three* circles. The following Table shows the elements which enter into the formation of each of these:—

[1] "Handbuch der Anatomie des Menschen."

[2] "Ueber die normale Krümmung der Wirbelsaüle." Müller's Archiv. 1854.

AXIAL LUMBAR CURVE.

No. of Spine in Plate II.	Segment 1.	Segment 2.	Segment 3.
No. 1 ♂.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2.	L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12.
No. 4 ♂.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2.	L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12.
No. 5 ♂.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2.	L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12.
No. 6 ♂.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2.	L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12.
No. 2 ♀.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2.	L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12 + D.V. 12.
No. 1 ♀.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2 + L.V. 2 + L.C. 1.	L.C. 1 + L.V. 1 + D.C. 12 + D.V. 12.
No. 4 ♀.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3 + L.C. 2.	L.C. 2 + L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12 + D.V. 12.
No. 3 ♂.	L.V. 5 + L.C. 4 + L.V. 4.	L.V. 4 + L.C. 3 + L.V. 3.	L.V. 3 + L.C. 2 + L.V. 2 + L.C. 1 + L.V. 1 + D.C. 12.

L.V. = Lumbar Vertebra.

L.C. = Lumbar Cartilage.

D.C. 12 = Intervertebral Cartilage between the last Dorsal and first Lumbar Vertebræ.

D.V. = Dorsal Vertebra.

From this Table it will be seen that in four of the male spines the same elements enter into the composition of each of the three segments, viz., the fourth and fifth lumbar vertebræ, with the intervening disc, forming the first segment; the second, third, and fourth lumbar vertebræ, with the two cartilaginous discs which separate them, forming the second segment; and the first and second lumbar vertebræ, with the disc between them, as well as the dorso-lumbar disc forming the third segment. No. 2 (female) agrees with these in almost every particular: the only difference being that in the highest segment the last dorsal vertebra is included. No. 1 (female) differs chiefly in the peculiar condition of the sacrum—a condition which excludes the first sacral vertebra from the sacral curve, and places it in the lumbar curve. The lumbar curvature in this spine therefore presents the segments of *four* circles—the abnormal segment being composed of the first sacral and last lumbar vertebræ, with the disc between them. The other segments correspond very closely with those of No. 2 (female), and differ only in the centre of the first lumbar disc constituting the point at which the two upper circles touch.

It will be readily understood that the determination of the composition of the different segments of the curve is one into which many fallacies may creep. A slight inaccuracy in the tracing, or the smallest inaccuracy in marking the central points of the vertebral bodies and intervertebral discs, or even a decided deviation from the mesial plane in sawing the spine, will tend to vitiate the results. No. 4 (female), and No. 3 (male), as will be seen from the Table, differ in a marked degree from the others, and the difference consists in a lengthening of the upper segment, at the expense of the intermediate segment. At the same time it should be noted that both of these spines offered peculiar difficulties in the investigation of this point. Certain of the vertebral bodies were, more or less, displaced in the antero-posterior direction, and showed, in consequence, an irregularity of position in relation to each other, which necessitated a mean curve being taken.

The question now is, Can we reasonably assume that the condition which is exhibited by Nos. 1, 4, 5, 6 (males), and No. 2 (female), is the

usual arrangement?—is this, in fact the composition of the normal lumbar curve? It may be so; but it appears to me that it would be rash, from the facts before us, to insist upon this point. It is vain for us to look for absolute uniformity in the composition of the curve of any region of the vertebral column.

Speaking of the composition of the lumbar curve, Professor Merkel says:—"We see directly that the arc of a circle comprising the whole lumbar column does not exist; but *such a one encloses the three middle lumbar vertebræ.* The first lumbar vertebra projects in front of this; the fifth retires behind it in a not inconsiderable manner; and if we wish to construct a curve which encloses the anterior surfaces of all the five lumbar vertebræ, it would probably approach most nearly a hyperbola." It is true that he refers here to the arc which is formed by the line joining the anterior projecting angles of the vertebral bodies; but it is interesting to note that in the four males mentioned above (Nos. 1, 4, 5, 6, Plate II.) the same vertebræ, with the intervening cartilaginous discs, are included in the second arc of the axial curve, and that in no case is it possible to introduce into this arc either the first or the fifth lumbar vertebra.

The three arcs which together built up the lumbar axial curve present radii of very different lengths, and in no two spines do the corresponding arcs exhibit radii of equal longitude. The lowest arc is a segment of the smallest circle, and it possesses a radius of an average length of 96·5 mm.: it varies in the different spines, however, between the limits of 63 mm. and 136 mm. The second, or intermediate arc, was found to possess an average radius of 341·7 mm: the longest radius exhibited by this segment was that of No. 4 (male), viz., 440 mm., and the shortest that of No. 4 (female), viz., 272·25 mm. The highest arc is the segment of a circle, so large that it is extremely difficult to arrive at an accurate conclusion as to the length of its radius. The spines under observation gave an average radius of 937·1 mm.; but in one case (No. 4, male) it was no less than 1732 mm., whilst in another (No. 1, female) it was only 555·08 mm.

Speaking generally, then, we may state that the average longitude of the radii of the three arcs which constitute the axial curve of the lumbar

column stand somewhat in the same relation to each other as the following numbers:

1 : 3 : 9

Professor Merkel, who likewise places the three middle lumbar vertebræ in the segment of one circle, believes that this arc is a very constant one. He states that "it has, in almost all the vertebral columns examined, a radius of 250 mm. . . . The remarkable fact, therefore, remains, that with absolute equality of the curves in the section of the vertebral column in question the smaller preparations have a relatively flatter curve than the greater." This uniformity of radius for the intermediate arc I have not only failed to obtain in my own tracings, but also in the reduced outline figures which illustrate Professor Merkel's Paper; and I may mention, that in the latter I have tested this point both by the examination of the axial curve, and also by the examination of the curve formed by the line joining the anterior angles of the vertebral bodies.

It is true that the average length of the radius presented by the *axial curve* of the three middle lumbar vertebræ of Professor Merkel's figures is 256 mm.; but it varies considerably in the different spines. The determination of the radius of the curve formed by the anterior faces of the same three vertebræ is a much more difficult matter, on account of the varying degrees of development of the vertebral bodies; but it appears to me that, whilst the average radius is very much that which has been given by Merkel, there are, nevertheless, considerable differences in this respect in the different spines.[1] Unless corresponding points are taken in each spine, from which to calculate the radius, the uniformity which results cannot be regarded as having any weight.

And now let us examine the results which have been obtained by Horner.[2] Although he deals with the curve formed by the anterior face of the column, it is, nevertheless, interesting to compare this with the

[1] Having submitted this difficulty to Professor Merkel, he was so courteous as to send me his original full-size drawings. An examination of these confirms the opinion which I have expressed above.

[2] Müller's "Archiv." 1854.

axial curve as obtained in the manner I have described. It must be borne in mind, however, that Horner, in his endeavour to arrive at the curvatures of the column as they exist in life, placed the spine under totally different conditions. He includes the two upper lumbar vertebræ in a section of the column which extends from the upper boundary of the ninth dorsal vertebra to the lower border of the second lumbar vertebra. The vertebræ lying within these limits he believes "form an almost completely straight line." The lumbar vertebræ III., IV., and V. constitute a second section, which represents the medium of movement between the lever-arm formed by the first section and the sacrum, and he compares it "to the carpal bones between the stiff metacarpus and the forearm."

It is possible that the anterior faces of the vertebræ which compose the first section may be bounded under certain conditions by a straight line; but it can be easily demonstrated that in no case do the axis lines of two contiguous vertebræ coincide: if prolonged, these lines invariably cut each other at an angle of greater or less magnitude. Merkel, in his criticism of Horner's work, very justly throws doubt upon the view that the greatest freedom of movement is obtained in the lumbar region below the second lumbar vertebra. He considers that the most unrestrained movement can be obtained rather at a point between the lumbar and dorsal vertebræ on the one hand, and between the lumbar vertebræ and the sacrum on the other. The fact that the point of inflexion is placed in the disc between the dorsal and lumbar regions favours this view.

But Horner further holds that the three vertebræ which he includes in the second section (viz., the third, fourth, and fifth lumbar), and which he considers essentially to be "the supporters of movement," are placed in such a position that they constitute the arc of a circle. Certainly, the axial line passing through them cannot be considered as such, and Merkel states that by his method he was not able to obtain such a curve for the anterior surface of this section of the spine.

Development of the Spinal Curves.—The statements which have been made upon this subject by the numerous observers who have studied the condition of the spinal form in the embryo and new-born child are at utter variance with each other. The writings of Horner, Parow, Cleland, Bouland, Balandin, Ravenel, and Charpy, deserve special attention.

To Horner the credit is due of being the first to carefully study the form of the vertebral column in the fœtus. The results of his investigations were published in 1854.[1] He says—"We know that the outline of the vertebral column in an embryo of a few weeks forms a straight line; that in the 5–6 months fœtus the projection of the promontory, the bending of the sacrum—which later possesses the most distinct curve—are still almost nil. Also the new-born child shows in its vertebral column relationships which deviate so distinctly from those of the adult, that already this difference is sufficiently great to show fully the significance of those factors (muscle activity and gravity), and the necessity of an investigation of their influence. But that the vertebral column of the new-born child is no longer straight can surprise no one who considers that muscle traction exercises its influence in the mother's body." He gives an outline of the vertebral column of a new-born child, and in this traces of all the curves are distinctly visible. In estimating the differences between this and the spinal form of the adult he makes the fatal error (as Balandin points out) of giving the pelvis of the new-born the same inclination which Meyer gives the pelvis of the adult in the erect attitude.

According to Parow[2] (1861), the dorsal concavity is the first curve which appears, and this is determined by the position of the fœtus in the uterus, and also by the connexion of the dorsal column with the thoracic walls. Even at this early date he appears to consider that the costo-sternal connexions keep the dorsal column in a state of tension. The accuracy of this view has been already discussed (p. 47). The cervical curve, he states, makes its appearance when the child raises its head; and with regard to the lumbar region he observes: "The already indicated convexity of the loin part of the new-born child must become more strongly

[1] Müller's "Archiv."

[2] Virchow's "Archiv."

pronounced under the influence of the weight of the head and upper part of the trunk."

In 1863, Professor Cleland, in a Paper communicated to the Biological Section of the British Association,[1] brought forward some entirely new and highly important points in connexion with the vertebral column of the new-born child. He demonstrated the influence which is exerted by the position of the lower limbs upon the form of the vertebral column, and the degree of inclination of the pelvis; and exhibited two drawings of mesial sections of new-born children. Of these, the one showed the condition of the vertebral column when the head is depressed upon the chest and the thighs are flexed, whilst the other represented the changes which are produced in the spinal form when the lower limbs are brought into a line with the trunk and the head raised. These tracings he has been so good as to place at my disposal. In the first, the vertebral column down to the sacrum forms one deep concavity; in the second attitude "the limbs could only be made to lie straight with the trunk by bending the pelvis back, so as to develop the lumbar convexity forwards of the column. Thus the straightening of the limbs when the child begins to walk is shown to be affected, not by mere motion of the hip-joint, but by development of the lumbar convexity of the vertebral column. In the drawing of the stretched body the brim of the pelvis is vertical; in the other drawing it is 52° removed from being in a straight line with the lumbar vertebræ immediately above it."

Dr. Bouland's Paper[2] upon this subject appeared in 1872. In estimating the curves in the new-born child he adopted a method of investigation which was altogether inappropriate for the purpose, and therefore his results cannot be regarded as possessing much value. At this period of its development the spine is extremely flexible: the slightest touch is sufficient to bend it in any direction, and therefore the most delicate manipulation is necessary to determine its natural curvature. Bouland removed the head, limbs, viscera, and soft parts, leaving only the thoracic walls and ligaments in position. He then, following the method of Weber, embedded the spine in plaster of Paris and divided it in the

[1] "Proc. Sect.," 1863, p. 112.

[2] "Journ. de l'Anatom. et Phys." Paris, 1872.

mesial plane with a saw. Infants of from five to six months, and children of three years old he treated in the same way. We need not be surprised, considering his plan of investigation, that his results, in so far as the lumbar region is concerned, were variable. In most cases the lumbar curve was absent both in the new-born child and in the infant of five or six months; but in certain cases it was observed to be established in both. The cervical and dorsal curves he found in a well-marked form at each of these periods. "During the first year the spine appears to retain the same configuration, but towards the end of the second year the lumbar curve commences to become more frequent. To start from this epoch, the vertebral column presents, in the majority of cases, the three curves which are present in the adult; at length, towards the fifth year, the spine presents all its antero-posterior flexures." His observations regarding the cervical curve are vitiated by the fact of his having removed the head. As we shall see further on, this flexure in the new-born child is entirely dependent upon the position of the head in relation to the anterior chest wall. The occasional lumbar curve, which he describes in the new-born child and five-months-old infant, was, no doubt, produced by a tilting back of the pelvis in the process of embedding in the gypsum.

Without doubt, the most important observations upon the development of the spinal curvatures are those by Dr. Balandin,[1] of St. Petersburg; and yet it must be admitted that he has brought forward little that is absolutely new. He confirms and extends the information which is given by Parow upon the origin of the cervical and dorsal curves, and corrects the views of this author in regard to the lumbar curve. Unfortunately, he was unacquainted with the observations of Cleland, which appeared ten years prior to the publication of his Paper in 1873; for he states, as a new and prominent fact—a fact, indeed, upon which a great part of his research hinges—the influence which the position of the lower limb exerts upon the moulding of the lumbar curve. The following is a brief account of the results which he obtained in the course of his very elaborate investigation.

[1] "Beiträge zur Frage über die Entstehung der physiologischen Krümmung der Wirbelsaüle beim Menschen." Virchow's "Archiv." 1873.

The dorsal curve is the first to make its appearance. It may be observed in an embryo of two months, but it does not consolidate until the fourth month. Balandin is in some doubt as to the causes which operate in the production of this primary curve, and gives, perhaps, undue importance to Parow's views in connexion with the relation of the thoracic walls to the spine. Much more likely it is due to the growth of the viscera. In the new-born and mature infant he holds that the cervical part of the column is straight, whilst the dorsal and lumbar portions, from the seventh cervical to the fourth lumbar vertebra, form a continuous concavity. At other times two concavities exist, viz., from the seventh cervical to the ninth or tenth dorsal vertebra, and a second flatter one from the tenth dorsal to the fourth lumbar vertebra. In some cases he observed the lower section to be in a straight line. The fifth lumbar vertebra "always exhibits a deviation towards the pelvis." It is a curious fact that he found the dorsal curve in the fœtus more stable than in the new-born child.

The same author believes that the origin of the cervical and lumbar curves is associated with the straightening of the body, from the bent position of the new-born child. Two stretchings take place at different periods, and in different situations. By the first the head is raised, the visual axis is brought parallel to the horizon, and the chin leaves the chest. This, as a rule, takes place at the third month, when the child is placed upright in the nurse's arms, and its result is the appearance of the cervical curve; at first this curvature is evanescent, but about the fifth month it consolidates.

The lumbar curve is the consequence of the second "stretching," which consists in the attempts which are made by the child to bring the trunk into a line with the thigh. This occurs at the end of the first year, or at the beginning of the second year, of life; or, in other words, when the child begins to make attempts at walking. The lumbar curve is long of consolidating. Balandin examined it in different subjects at the tenth, twelfth, sixteenth, and twentieth year, and found that by stretching the column he could cause the convexity to disappear. He considers, therefore, that it does not become absolutely stable until adult life.

But how is it that, when a dead newly-born child or a fœtus is placed on its back, the forcing of the thighs into a line with the trunk produces so marked a change in the inclination of the pelvis, and at the same time leads to a convexity forwards of the lumbar part of the column? Balandin goes into this point very fully, and he comes to the conclusion that the factor at work is the ilio-femoral ligament. He asserts that when this structure is divided on both sides, no movement of the limbs will alter the position of the pelvis or the form of the lumbar column. In moving the fully-flexed thighs of an infant in a recumbent attitude, so as to bring them into a line with the trunk, the first part of the movement occurs at the hip-joints; soon, however, the extension is brought to a close, by the ilio-femoral ligaments: the femur and pelvis now move *en bloc*, and the spine becomes curved in the lumbar region.

It is a very difficult point to decide whether the ilio-femoral ligament in the child is relatively shorter than in the adult. With the ligament intact, the thigh of the child has apparently the same range of movement at the hip-joint as the thigh of the adult. But then, in the combined movement of thighs and pelvis, which is required to bring the lower limbs of the child into a line with the trunk, the pelvis assumes a greater degree of obliquity than in the erect adult. As Cleland has shown, the brim of the pelvis in this forced attitude is vertical, or nearly so; further, the artificial lumbar convexity which is produced is more prominent than the natural curve in the adult in the proportion of 11 to 9. (Plate III. female fœtus.) This, perhaps, might be regarded as an indication that the infantile ligament is relatively slightly shorter: although it is just possible that the hitherto unstretched ilio-psoas muscle may also exercise an influence in this direction. At the same time it is obvious that the relative range of movement which exists at the hip-joint in the infant and the adult, and the agents which influence this, can only be satisfactorily ascertained by a study of the growth of the hip-joint. The innominate bone, and the relation of the head and neck of the femur to its shaft, undergo marked changes in their transition from the infantile to the adult condition.

Ravenel,[1] in his measurements of the spine, comes to the conclusion that "the infantile vertebral column is a neutral form, which later on is differentiated in a specific way". In three newly-born children he found that the anterior and posterior faces of the different regions of the column were equal in length, and that consequently the new-born spine is devoid of curvatures of any kind. We cannot place much reliance upon these measurements, when we remember that the pliability of the flexible infantile column was exaggerated by the removal of the neural arches. If it were examined upon a flat surface, we need not be surprised that the measurements of its two faces presented equal results.

Charpy[2] has given utterance to the most recent views on the development of the spinal curves, and these are somewhat startling. He holds that in the four-months fœtus the dorsal curve is alone present; that in the last months of gestation the cervical curve becomes more and more manifest, and that at the eighth month the lumbar curve may be recognised. In spite of the "doubled-up" position of the fœtus in the uterus all the curves are developed before its birth. He says: "I have always in the new-born child verified the existence of the lumbar curve. It is, perhaps, very feeble, but it is constant; its 'flèche' may be scarcely measurable, but a thread stretched vertically renders the arc manifest to the eye". Further, he considers that the lumbar curve remains rudimentary until the child begins to walk, and that the type becomes fixed towards the fifth year. He is evidently unacquainted with the important observations of Cleland regarding the effect produced on the infantile lumbar column by the straightening of the thigh; and we shall see later on that the comparison which he draws between the human vertebral column, at different stages of its development, and that of the quadruped and anthropoid, is altogether untenable.

Enough has been said to show how varied are the views of those anatomists who have dealt with the development of the vertebral curves.

[1] "Die Maasverhältnisse der Wirbelsaüle und des Rückenmarkes beim Menschen." Zeitschrift für Anat. und Entwicklungsgeschichte. 1877.

[2] "Journ. de l'Anatomie et Physiologie." Paris, 1885.

This diversity of opinion has undoubtedly arisen from a failure to recognize the important effect which the position of the limbs and the head exerts upon the pliable column of a fœtus or mature infant. My own work in this field has been limited to the examination of the embryo at different stages of growth and to the newly-born child. The difficulty of obtaining material rendered it impossible for me to push my inquiries further. Still the field was wide enough to enable me to arrive at very definite conclusions as to the order and manner in which the curves appear, and also as to their relation to the intra- and extra-uterine periods of life. The method which I employed for the three new-born infants which I examined was that of freezing the entire body, and then making a mesial section by means of a fine saw. In each case the child was frozen on its back, with its head and limbs in a given position. The embryos were hardened in spirit, and then divided with a sharp knife in the mesial plane.

In the very young embryo the vertebral column, or, I should rather say, the parts which represent it at this early stage, are bent in the form of a single arch, with the concavity to the front. The curvature in the lower portion, which corresponds to the sacral and the coccygeal regions, is more abrupt than above, and thus the extremity of the segmented column is turned up in front of the lumbar region like a hook. This condition can be seen in the beautiful drawings which are given by Professor His in his work entitled, "Anatomie menschlicher Embryonen." A second curve soon makes its appearance, and this is accomplished by the bending back of the sacral and coccygeal portions of the column. The vertebral column now presents two curves—both concave to the front—viz. an upper curvature, which involves the cervical, dorsal, and lumbar regions, and a lower sacro-coccygeal concavity. These increase in depth, but no further flexure of the spine takes place so long as the fœtus is retained *in utero*. Fig. 7, D (p. 76) represents a full-sized tracing of an embryo, 46 mm. long, in which both of these flexures are exhibited. The long upper concavity, according to Gegenbaur,[1] may be looked upon "as an adjustment to the less stretched-out ventral parts of the

[1] "Lehrbuch der anatomie des Menschen."

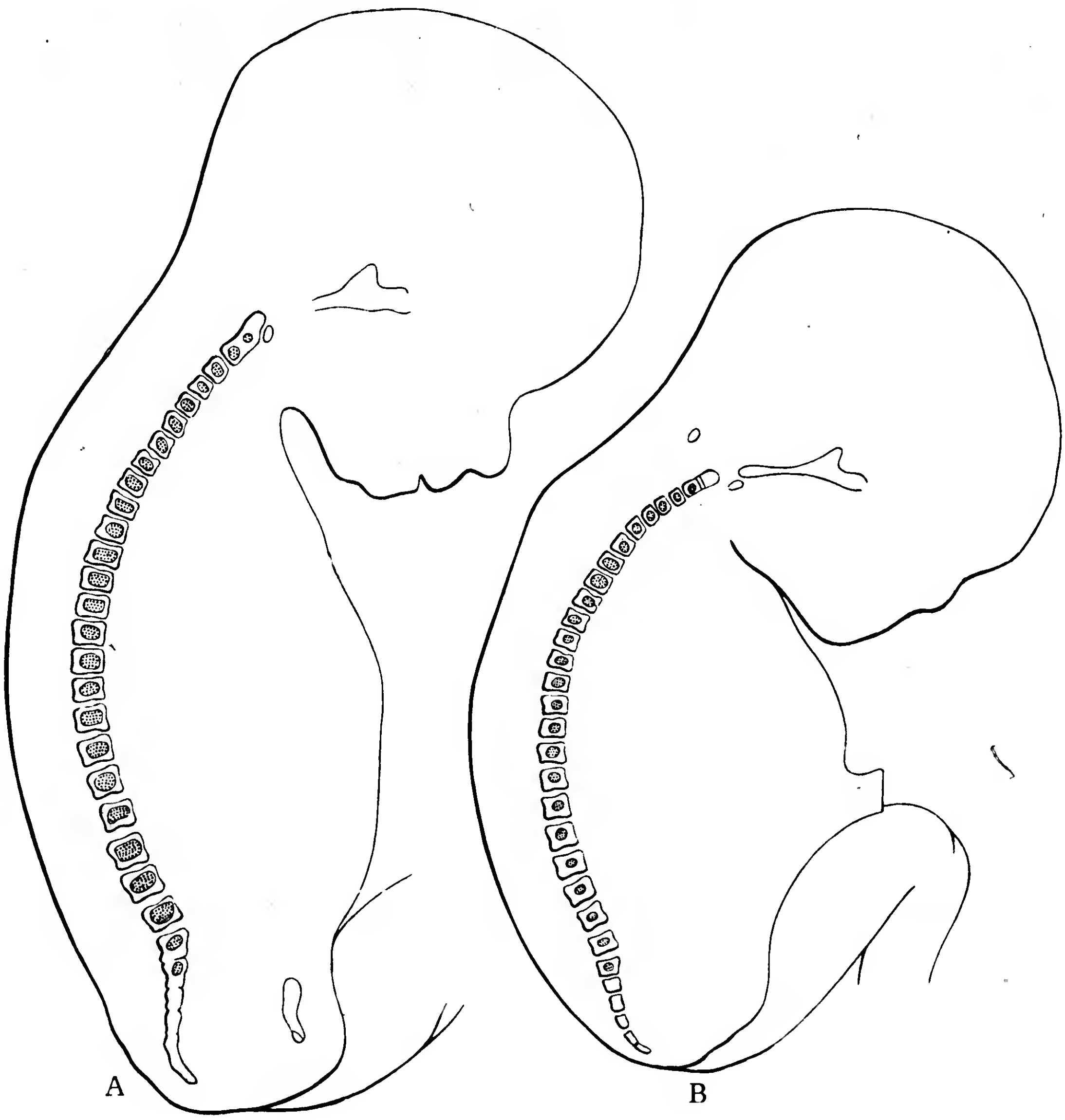


Fig. 6.

Full-size tracings of the vertebral column of two embryos, of different periods of growth, cut in the mesial plane :—

A. Embryo, ... 153 mm. in length.

B. Embryo, ... 122 mm. in length.

In both the promontory is evident and the two initial curvatures apparent.

body," and no doubt it is also partly due to the growth of the viscera. Later on it is maintained by the position which the fœtus assumes *in utero.*

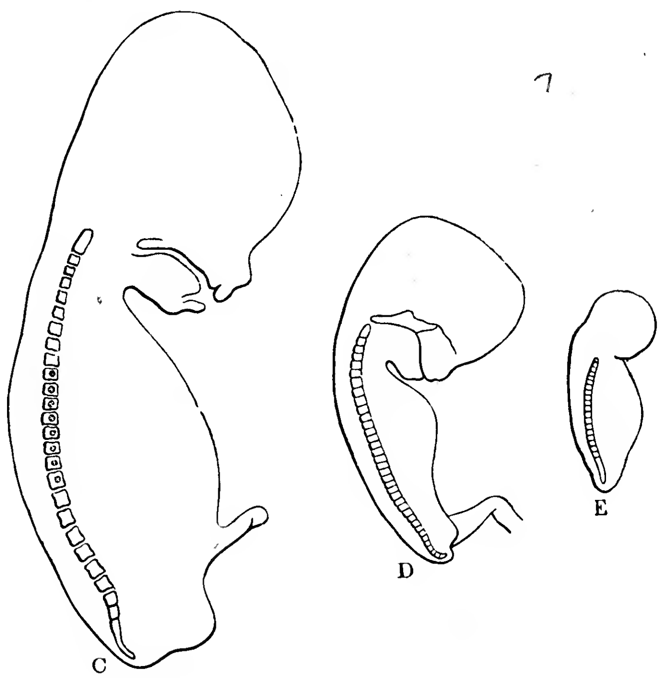


Fig. 7.

The woodcuts represent full-size tracings of mesial sections of three embryos of different periods of development:—

C. Embryo, 87 mm. in length: the promontory is quite apparent, and the two concavities of the column very evident.

D. Embryo, 46 mm. in length: two curvatures are visible, but the promontory is just beginning to make its appearance.

E. Embryo, 27 mm. long. The long upper curve is present, but not the sacral. Still, even here, a tendency to a backward inclination of sacrum is seen. The section was not very successful. The specimen was one which I found in the stores of the Trinity College Museum, and it had not been hardened properly. It was, therefore, impossible to obtain an accurate outline of the body. The form of the vertebral column, however, is fairly well reproduced.

The demarcation between the sacrum and the upper part of the vertebral column is foreshadowed at a very early period—very much earlier indeed than Horner would have us believe. A feeble, but distinct,

promontory, caused by the bending back of the sacrum from the line of the rest of the column, is evident in embryos of 70 mm. long. But even before this stage of development the backward sweep of the pelvic part of the column can be detected: in these early stages, however, the transition is gradual, and no promontory is visible. This can be seen in fig. 7 D, which represents an embryo 46 mm. long; and even in fig. 7 E, which shows the spinal form of an embryo only 27 mm. long.

In the new-born child, when it is laid upon its back with its head flexed so that the chin rests upon the chest, and with the lower limbs lying easily and without restraint, two concave curves are alone visible, as can be readily proved by freezing an infant in this attitude, and then dividing it by a saw in the mesial plane. The first curve, which extends from the summit of the spine to the last lumbar vertebra, shows a gentle and continuous concavity to the front; the second curve is the sacral curve, and both are sharply marked off from each other by a feeble promontory. It is true that I have only examined one child in this posture, but I am confirmed in my views by one of the tracings which Professor Cleland has so kindly put into my hands. This represents the mesial section of an infant in a similar attitude. The only difference is, that in his drawing he represents the dorsal concavity as being deeper than I have seen it. This is probably due to the fact that he examined the child while placed upon its side. Balandin states that in all cases he found the fifth lumbar vertebra deviated towards the sacrum. I have not observed this, and certainly it is not the case in Cleland's drawing.

If the head of the child is now raised so as to cause the chin to leave the chest, a change of form in the vertebral column is seen to take place in the cervical region.[1] This corresponds to "the first stretching" of Balandin. The cervical column now shows a gentle convexity to the front. Fig. 1, Plate III. represents the reduced tracing of the mesial section of a child frozen in this attitude. The dorso-lumbar concavity still remains: the only change is in the neck. As we have seen, Balandin fixes the date of the first appearance of this curve at the third month, and con-

[1] This can also be seen in the young embryo. Raise the head, and immediately a cervical curve appears.

siders that it consolidates in the fourth and fifth months. It is true that a child does not raise its head for a short time after birth, but still the curve will occur in a transient manner every time its head falls over the arm of its nurse.

In Plate III., Fig. 2, a reduced tracing is given of the mesial section of a child which was frozen on its back, and with its thighs forcibly extended, and retained in this position by long pins thrust through the knee-joints so as to nail them closely to the board upon which the infant was lying. The head was allowed to assume the position most natural to it, and this was a position in which the chin rested upon the chest. The cervical and dorsal columns form one continuous concavity to the front, whilst the lumbar region is arched forwards in a most remarkable way. This attitude represents, therefore, the "second stretching" of Balandin—a stretching which takes place when the child begins to make attempts at walking. Reference has already been made to the probable factors at work in producing this extraordinary tilting back of the pelvis, and consequent arching forwards of the lumbar column, so we need not enter further upon this point. It is certainly a very beautiful provision for the determination of the lumbar curve in the proper direction.

Cleland's representation of the mesial section of a child in this attitude shows a position of the pelvis almost identical with that figured in Plate III., but the lumbar column is not bulged forwards to the same extent.

Balandin is probably right as to the period at which the lumbar curve consolidates. Certainly Bouland and Charpy give for this a much too early date. Dr. Symington, at the meeting of the British Association held in Aberdeen (1885), exhibited drawings of mesial sections of a boy of six years old, and a girl of thirteen years old, and in both of these the curve is feebly marked. Although his work upon "The Topographical Anatomy of the Child," is not yet published, he has kindly allowed me to introduce reduced tracings of the spines as exhibited in his drawings (Plate III. girl, *aet.* 13; boy, *aet.* 6).

In the forced lumbar curve exhibited in Plate III. (female child), the dorso-lumbar point of inflexion is in the intervertebral disc between the two

regions. Balandin remarks that the younger the fœtus the higher is this point of inflexion, and the greater, therefore, is the portion of the dorsal region which is involved in the lumbar curve.

From what has been said, we may therefore conclude that the cervical and lumbar curves are secondary, and that they are first called into existence by muscular action: the muscles which raise the head give the initiative to the cervical curve, and the great mass of the multifidus and the erector spinae, which draws the upper part of the trunk backwards into a line with the thigh, when the child first begins to make efforts at standing erect, performs a like office in the case of the lumbar curve.[1] Of course in producing the artificial lumbar curvature, which is figured in Plate III. (female fœtus), by the straightening of the lower limbs, the process is the opposite of what takes place in life, inasmuch as it is the pelvis that is tilted backwards upon the spine by the bending of the lumbar region, and not the spine upon the pelvis: in both cases, however, the result is the same.

The relative Length of the Lumbar portion of the Spine.—Luschka[2] and Hyrtl,[3] both state that the lumbar portion of the vertebral column is relatively somewhat longer in the female than in the male; but they do not enter into any particulars as to how they have established this fact, nor do they give the amount of the difference between the two sexes. Having tested the accuracy of the statement by measurements instituted upon the frozen spine, I am in a position to corroborate it. The following Table gives the relative lengths of the three regions of the spine, taking the standard 100 as representing the total length of the vertebral column when measured along its fore-surface.

[1] Merkel believes that the lumbar curve is produced by the unequal growth of the neural arches and vertebral bodies. The rate of growth of the bodies and intervertebral discs is much more rapid than that of the arches, and as a consequence the lumbar convexity is developed. As we have seen, the explanation of the manner in which this curvature springs into existence is much more simple.

[2] "Die Anatomie des Menschen."

[3] "Lehrbuch der Anatomie."

RELATIVE LENGTH OF DIFFERENT REGIONS OF THE ADULT VERTEBRAL COLUMN. FORE-SURFACE OF ENTIRE COLUMN = 100.

MALES.				FEMALES.			
No. of Spine in Plate II.	Cervical region.	Dorsal region.	Lumbar region.	No. of Spine in Plate II.	Cervical region.	Dorsal region.	Lumbar region.
No. 1,	22·2	45·4	32·4	No. 1,	21·2	46·9	31·9
No. 2,	22·2	46·3	31·4	No. 2,	21·1	45·8	33·1
No. 3,	23·1	45·7	31·2	No. 3,	21·9	43·8	34·3
No. 4,	21·3	46·9	31·8	No. 4,	20·6	46·6	32·8
No. 5,	21·6	46·8	31·6	No. 5,	23·0	46·0	31·0
No. 6,	20·0	48·0	32·0	—	—	—	—
Average,	21·8	46·5	31·7	Average,	21·6	45·8	32·8

The average difference in relative length of the lumbar region in the males and females is therefore—

31·7, males,
32·8, females.

And the increased length in the female is accompanied by a corresponding shortness of the dorsal region, viz.,

46·5, males,
45·8, females.[1]

If we measure from the central point of the upper surface of the first sacral vertebra—(*a*) to the central point of the upper surface of the first lumbar

[1] Since writing the above I observe that Aeby, in his Paper upon "Die Altersverschiedenheiten der menschlichen Wirbelsaüle" (Arch. für Anat. und Entwick, 1879), although he states in his text-book that he could find no essential difference in this respect between the male and female, gives the measurements of thirteen adults of both sexes, and these agree in the most remarkable manner with the results in the above Table. The following are his figures:—

	Lumbar region.	Dorsal region.
Males,	31·3	46·6
Females,	32·8	45·7

vertebra, and then—(*b*) to the tip of the odontoid process, and compare the measurements obtained, we find for the spines figured in Plate II. the following results:

Males,	30·3
Females,	31·0

The relative difference in the length of the lumbar segment of the spine in the male and female is in this case somewhat reduced, and this reduction is due to the greater degree of curvature in the female.

Topinard,[1] from measurements of a large number of "colonnes montées et munies de leur disques," gives the relative length of the dorsal region to the lumbar region as 3 to 2. This is a fairly accurate estimate, and it speaks well for the skill with which the skeletons he examined are mounted.

In the newly-born child the proportions which are exhibited by the different regions of the vertebral column are very different from those of the adult spine. The lumbar region is relatively much shorter, whilst the cervical and dorsal regions are both relatively longer. Aeby was the first to point out this peculiarity of the infantile spine, and he gives the following figures, based on the measurements of five new-born children:—

Cervical region.	Dorsal region.	Lumbar region.
25·6	47·5	26·8

The results which I have obtained by measuring the frozen spines of three infants are almost identical with those of Aeby, viz.:—

Cervical region.	Dorsal region.	Lumbar region.
25·1	48·5	26·4

The following Table gives the proportions of the different regions of the spine in three fœtuses of different periods of development:—

[1] "Des anomalies de nombre de la colonne Vertébrale." Revue d'Anthrop. Paris, 1877.

RELATIVE LENGTH OF DIFFERENT REGIONS OF SPINE IN THE FŒTUS.
FORE-SURFACE = 100.

	Cervical region.	Dorsal region.	Lumbar region.
A.—Fœtus 153 mm. in length, as measured from the summit of the head to the opposite extremity of trunk,	26·8	47·4	25·8
B.—Fœtus 122 mm. in length, measured in same manner as A,	23·4	50·6	26·0
C.—Fœtus 87 mm. in length, measured in same manner as A,	24·4	47·3	28·3

From this it would appear that the lumbar region in the very young fœtus has a proportionate length, which approaches more closely to that of the adult than that of the more fully developed fœtus or the new-born child. Further investigation, however, would be necessary to confirm this point. Still, the bodies of the lumbar vertebræ at this early period, so far as the eye can judge, seem long and narrow in comparison with those of more advanced specimens: they present a shape, indeed, which recalls somewhat the appearance of the lumbar vertebræ of certain of the lower apes.

Anthropoid Apes.

The condition of the lumbar column in the anthropoid apes must now claim our attention. Most observers deny the existence of the lumbar curve in all animals, save man.

Goodsir,[1] in his lectures upon "The Dignity of Man," states that "in no form of animal vertebral column are the secondary curvatures so highly developed as in the human, and no animal possesses the lumbar curvature." Sir William Turner[2] is equally positive, and asserts that "in the human spine alone are the lumbar vertebræ convex forwards." Sir Richard Owen expresses no opinion upon this point, in so far as the chimpanzee and gibbon are concerned; but he denies the presence of the curve in the gorilla, and in the orang. With regard to the first he says:—"The whole series of true vertebræ in the gorilla form but one curvature, which is slightly concave forwards, especially in the dorsal region;[3]" and of the orang he remarks that "the entire vertebral column has one general curve dorsad, from the altas to the commencement of the sacrum, where there is a slight curve in the contrary direction.[4]"

According to St. George Mivart[5] "the beautiful sigmoid curve formed by the dorsal and lumbar vertebræ of man exists in no other species; but the nearest approximation to it is found not in the highest apes, but in the *Cynocephalus*." He repeats this statement in his treatise entitled "Man and the apes." Professor Macalister[6] is of opinion that "the curves in the spine of the apes exactly correspond to the curves in the spine of an exceedingly young human infant, and they never assume the adult human curve."

[1] "Anatomical Memoirs of John Goodsir": edited by Turner.
[2] Introduction to "Human Anatomy," vol. i.
[3] "Comparative Anatomy and Physiology of Vertebrates," vol. ii.
[4] "Osteology of the Chimpanzee and Orang-utan."—*Trans. Zool. Soc.* 1835.
[5] "Axial Skeleton in the Primates."—*Proc. Zool. Soc.* 1865.
[6] "Discourse delivered before the Royal Dublin Society." 1862.

Professor Aeby[1] discusses this question at some length; but the premises upon which he proceeds are utterly erroneous. It would be well to quote his own words. He says:—"The curvatures of the human vertebral column are, it is known, the result of its elasticity, when under weight. In time, however, they lead to a permanent transformation of the vertebral bodies, and, as I have already shown elsewhere, by measurement, they effect a wedge-like diminution of these towards the concave side. The process is a mechanical one, and there is no doubt but that the laws by which it is governed are of equal force when applied to every vertebral column as well as to the human one. On these premises I maintain that in the case of the spinal column of the gorilla the form of the vertebral bodies permits us to draw a conclusion as to the spinal curves, and also to determine the direction in which the column is bent. . . . The difference in height of the vertebral bodies in man and the gorilla are so great and so characteristic that their value is of typical interest. In man the wedge-like diminution in front extends from the fifth to the twenty-first; in the gorilla from the sixth to the twenty-fourth vertebra, *i. e.*, in man from about the middle of the neck to the middle of the lumbar region; in the gorilla, however, to the lower end of the lumbar region. The lower true vertebræ, therefore, preserve with strict exactness the characters of their fellows higher up, whereas it is just the opposite in man. The difference is too manifest to allow the contradictory proof of some individual case, or to expect a compensation from the shape of the intervertebral substance. In other words, the characteristic curvature of the loins in man is wholly wanting in the gorilla. Its vertebral column, as Owen has shown, extends in an uninterrupted curve concave to the front down to the sacrum; and Huxley is surely wrong when he gives it a curvature similar to that of man." Aeby likewise gives the measurements of the lumbar vertebræ of two gibbons, and comes to a similar conclusion in respect to these members of the anthropoid group.

We have already seen enough to be aware that the argument which Professor Aeby uses in support of his assertion that the gorilla is destitute

[1] "Beiträge zur Osteologie des Gorilla."—Morph. Jahr., vol. iv.

of a lumbar curve is utterly valueless. Many of the lower races of man show a condition of vertebræ closely analogous to that of the gorilla, and we know that in certain of these races the lumbar curve, so far from being absent, is indeed exaggerated. Further, the vertebræ of the chimpanzee are much more wedge-shaped (with the broad end backwards) than the vertebræ of the gorilla, and, as we shall see later on, the former animal possesses a very evident lumbar convexity. In short, everything goes to prove that the intervertebral discs alone are quite sufficient of themselves to build up a lumbar convexity, even in cases where the vertebral bodies are shaped in a fashion antagonistic to the curve. I am even inclined to believe that once the direction of the curve is determined in such a case, the greater amount of elastic material in the column is likely to lead to an exaggeration of the curvature. This would account for the increased lumbar curve in the Hottentots and the other low races in which it has been stated to exist.[1]

Dr. Adolf Pansch[2] in his lectures to advanced students, makes a very decided assertion regarding the lumbar curve. He calls it "the characteristic badge of the upright human attitude," and he goes on to say that if we investigate the matter even in the gorilla and chimpanzee, when they walk upon their lower limbs, we will convince ourselves that "they have no proper loin curve, and also that they have no proper upright walk."

And now let us look at the statements that have been put forward upon the opposite side of this question. Three anatomists, viz. Huxley, Broca, and Topinard, have advocated the presence of a lumbar convexity in the anthropoid ape; but in several points these observers are not quite in accord with each other in this matter.

Huxley[3] was the first to give definite information regarding the lumbar curve in the anthropoid apes. In 1863 he writes as follows:—"The vertebral column of the gorilla as a whole differs from that of man in the less marked character of its curves, especially in the slighter

[1] The same might account for the very distinct *ensellure lombo-sacrée* in a child when standing erect—a condition which disappears at once when the child assumes the horizontal position.

[2] "Anatomische Vorlesungen."—Theil, i. 1884, p. 44.

[3] "Man's Place in Nature." 1863.

convexity of the lumbar region. Nevertheless the curves are present, and are quite obvious in the young skeletons of the gorilla and chimpanzee, which have been prepared without the removal of the ligaments. In young orangs similarily preserved, on the other hand, the spinal column is either straight, or even concave forwards throughout the lumbar region." In his work upon the "Anatomy of vertebrated animals,"[1] he compares the dorso-lumbar portion of the column of the orang to that of a new-born child, and repeats his statement as to the correspondence of curvature in the gorilla, chimpanzee, and man.

But the examination of skeletons, whether natural or artificial, is not calculated to convey absolutely correct information upon a matter so difficult to appreciate as the exact amount of spinal curvature in a given animal; and, therefore, we need not be surprised that Huxley makes the remark that "the question of the curves of the column in the apes requires further investigation."

Broca differs from Huxley in so far that he denies the presence of a lumbar convexity in the gorilla. The dorso-lumbar region, according to this authority, forms a continuous concavity; or the last two lumbar vertebræ may be straight, and be related to the column above in the same way that "the handle of a vine-knife is related to its blade." In the orang he limits the lumbar curve to the last lumbar vertebra, and in the chimpanzee to the lower two lumbar vertebræ: above that level in each the lumbar vertebræ participate with the dorsal vertebræ in the formation of a continuous concavity. Further, Broca states that among the anthropoid apes the gibbon possesses the most perfect lumbar convexity—the saimang only differing from the human type in the degree of curvature. Indeed, the convexity occupies the entire extent of the lumbar region. In the Hylobates agilis, and in the gibbon of Raffles, the lumbar curve is not quite so strongly marked, but it succeeds the dorsal curve at the same level.

Topinard, in his work on Anthropology, follows Broca closely in the views which he has expressed upon the spinal curvature of the anthropoid apes.

[1] 1882. p. 404.

The Gorilla.—The only evidence which I have to offer regarding the lumbar curve in the gorilla is that which is afforded by an examination of the macerated vertebræ. I found it impossible to obtain the fresh spine of a specimen of this ape.

The lumbo-vertebral index of the gorilla approaches more closely to that of man than the corresponding index of either the chimpanzee or the orang. Indeed in this respect it is only removed from the lower races by the high index of the last lumbar vertebra. On this account I am inclined to believe that in the gorilla the lumbar convexity must be highly developed. It is true that we have noticed that the lumbo-vertebral index is by no means a sure and safe guide to follow in drawing inferences in regard to the curvature of the lumbar region; but that it has some value is shown by the fact that in the European spines the lumbo-vertebral index was found to have a general correspondence with the degree of convexity. Moreover, in the gorilla the last lumbar vertebra approaches more closely the shape of that of man than the corresponding vertebra in any of the other anthropomorphic apes, and this appears to me to be a fact of some significance. Still, in view of the opposite and contradictory statements of Broca, Owen, and Huxley, it must be admitted that the question of the lumbar curve in the gorilla can only be satisfactorily determined by the examination of a fresh specimen by the method I have adopted for the other apes.

Chimpanzee.—In the prosecution of this investigation I have been so fortunate as to secure four chimpanzees very shortly after their death. The two largest specimens—a male and a female—I purchased from Mr. Cross of Liverpool; another female I received from the Council of the Zoological Society in Ireland, whilst the remaining specimen was presented to me by Dr. William Fraser of Dublin. None of these specimens had attained maturity.

Three of these chimpanzees I divided in the mesial plane with a saw, after they had been thoroughly frozen. In each case the animal was placed on its back in a zinc box. Great care was taken to have

it properly straightened, so that the vertebral column might be cut exactly in the mesial plane in all its length. By driving a long pin into the interval between the two central incisor teeth of the upper jaw, a second pin into the middle of the sternum, and a third into the centre of the symphysis pubis, and stretching a string between the projecting parts of the pins, the body could be very accurately adjusted. No support whatever was given to the back, and the limbs were allowed to lie naturally, and without restraint of any kind. The latter precaution, as we shall see later on, is a matter of great importance. The box was now placed in the freezing mixture, and it was found to be necessary to allow it to remain in this for about seventy-two hours to obtain a satisfactory result. The bushy hair of the animal tended very much to delay the process of freezing.

The section of the male was very successful; the vertebral column was cut as nearly as possible in the mesial plane, along its whole length. In plate III. (Troglodytes niger; male No. 1), a reduced tracing of the cut surface is given. Figure 3, plate IX., which is taken from a photograph of the animal after it was cut, and while still in the frozen condition, shows the line of the section, and also the position of the trunk and limbs. In front the section is not accurately in the middle line, but the deflection is so slight that it did not materially affect the section of the vertebral column. Plates VII. and VIII. are full-size coloured lithographs, which were prepared from tracings of the cut surfaces of the two sides of the section, while the body was still in the frozen state.

The following were the dimensions of this chimpanzee:—

Length of trunk, ...	$20\frac{1}{4}$ inches.
From the summit of the head to the sole of the foot, when the lower limb was extended as far as it would go without using force (the thigh remained partially flexed), ...	$28\frac{1}{2}$ inches.

The milk teeth were still in place. In all probability it was between three and four years of age; but this is a point which is extremely difficult to determine.

The female, which in Plate III. follows the male (Troglodytes niger; female No. 1), was apparently a little younger; its entire length was $27\frac{1}{2}$ inches, whilst its trunk measured $19\frac{1}{2}$ inches. Unfortunately, in placing the box which contained it into the freezing mixture, the animal was allowed to shift its position; it was therefore found impossible to cut the entire length of the vertebral column in the mesial plane. Sufficient, however, is divided for us to determine the degree of lumbar curvature.

The third specimen was a very young female, which was rescued from the hands of a bird-stuffer by Dr. William Fraser. When it came into my possession it was completely flayed, and its head and limbs had been removed. A very successful section of the trunk was obtained, and the reduced tracing of the cut surface is given in Plate III. (Troglodytes niger; female No. 2). The length of the headless trunk was only 14 inches; so that it is not likely that this animal was more than two years old.

The last specimen was a large female, 32 inches long. None of the milk teeth were shed, and I believe that it must have been between four and five years old when it died. This animal I did not consider it necessary to freeze. Having removed the head and viscera, and cleaned the anterior face of the vertebral column, I obtained a tracing of the spinal curvature by means of a strip of ductile lead. This, on a reduced scale, is given in Plate III. (Troglodytes niger; female No. 3).

All these specimens came into my possession within twenty-four hours after death, so that they were examined in a perfectly fresh condition.

A glance at the four tracings in Plate III. will be sufficient to show that the spinal curvature of the chimpanzee, is very similar to that of man, and what difference there is is chiefly to be found, not in the secondary curves, but in the primary dorsal and sacral curves. These are not so pronounced, and owing to the small degree of sacral obliquity the promontory is altogether wanting.

Points of Inflection.—The point at which the dorsal and cervical axial curves meet is the same as in man, viz. the central point of the intervertebral disc, between the 2nd and 3rd dorsal vertebræ. In the male chimpanzee (Troglodytes niger; male, Plate III.), the point of demarcation, between the dorsal and lumbar axial curves, is the central point of the intervertebral disc, between the 12th and the 13th dorsal vertebræ; whilst in female No. 2 it is placed a little higher, viz. in the body of the 12th dorsal vertebra. Here again there is a close correspondence with what we have already noticed in the human vertebral column. The lower limit of the lumbar curve is different. Owing to the feeble obliquity of the sacrum, the first sacral vertebra is included in the lumbar curve in three of the specimens, whilst in the young female (No. 2) the two upper sacral vertebræ enter into the construction of the curve. This constitutes the essential difference between the lumbar curve of man and that of the chimpanzee ; the upper limit of the curve is the same in both; but, owing to the absence of the promontory, one or two of the sacral vertebræ lie in the same line of axial curvature. But an approach to a similar arrangement is occasionally met with in the human subject. In Plate II. the spine of a female (No. 1) is depicted, in which the first sacral vertebra is included in the lumbar curve. Further, in the tracing of the boy (*æt.* 6.) in the same Plate, it will be noticed that the first sacral vertebra occupies a position which brings it very nearly into the line of the axial lumbar curve.

Degree of Lumbar Curvature.—Owing to the greater proportionate length of the lumbar curve in the chimpanzee than in man, it is difficult to compare the degree of curvature in each. Still if we exclude the sacral part in the chimpanzee, and strike an index of curve, as we have already done in the case of the human spines, for that portion which is composed of the lower five true vertebræ, we shall obtain sufficiently accurate results. This index, which expresses the degree of curvature of the anterior face of this section of the column, is calculated, as we have already seen, by taking the length of the section composed of the five lower true vertebræ (measured from the centre of the upper surface of

the thirteenth dorsal vertebra to the centre of the lower surface of the fourth lumbar vertebra) as the standard, and equivalent to 100, and then comparing it with the distance between the intersecting line (see tracings, Plate III.) and the point of greatest prominence. The most projecting point in the male and in the young female (No. 2) was the upper border of the third lumbar vertebra (corresponding in this respect with the male human vertebral column), whilst in female No. 1 the most projecting point was the lower border of the second lumbar vertebra, as in the human female. The following indices of curve were obtained:—

Male Chimpanzee, ...	7·1
Female Chimpanzee, No. 1, . .	8·7
Female Chimpanzee, No. 2, . .	10·0
Female Chimpanzee, No. 3, . .	9·2

These give an average index for the four specimens of 8·7, which is very nearly the same as we obtained for the seven adult male human spines figured in Plate II. (viz. 8·8), and tabulated in Table R., p. 27.

These results are very extraordinary, when we remember that none of the chimpanzees which were examined had reached maturity; indeed it is very questionable whether the oldest of them was more than four and a-half years old. It is still further strange that the youngest female (No. 2), which certainly was not more than two years old, should have the strongest lumbar curve (index 10). In the spine of the boy, figured in Plate III. (*æt.* 6), the curve index is 4·1; and in the girl (*æt.* 13) it is 6·4.

From these facts, therefore, we are justified in concluding—(1) that the lumbar convexity in the chimpanzee is apparently as pronounced as in the adult human male; and (2) that the lumbar curve is established at a very much earlier date. I am not aware if it is known at what age a chimpanzee reaches maturity; but there cannot be a doubt but that its growth and development proceeds more rapidly than in the human child. At two years old a chimpanzee is more advanced than a child of the same age; and this, no doubt, is the reason of the earlier development of the lumbar convexity.

It is well known that the chimpanzee does not possess the power of fully extending the thigh at the hip-joint. The limit of downward movement of the thigh is reached when the femur forms with the axial line of the trunk an angle of 131° open to the front; but if force be employed, the limb can be brought into the same line with the trunk: in this case, however, the movement is not at the hip-joint; the thigh and pelvis move together, and the bending takes place in the lumbar region of the spine, and the consequence is a great increase of the lumbar convexity. In performing this experiment on female No. 1, after it had been divided in the mesial plane, I found that not only did the convexity become more pronounced, but that when the thigh reached a point where its axis line corresponded with that of the trunk, the lumbar curve had involved the dorsal region, as high as the 10th dorsal vertebra. This is interesting when we remember that in the new-born child the bringing of the thigh and trunk into the same line is attended with very similar results.

We may summarize the foregoing facts regarding the spinal curvature of the chimpanzee thus:—

(1). The dorsal curvature is flatter than in man.

(2). There is no promontory, and a very small degree of sacral concavity.

(3). The axial dorsal curve is limited above by the central point of the intervertebral disc between the 2nd and 3rd dorsal vertebræ, and below by the central point of the disc between the 12th and 13th dorsal vertebræ, or by the central point of the 12th dorsal vertebra.

(4). The lumbar curve includes one or two of the sacral vertebræ.

(5). The lumbar convexity is developed at a very early period, and is probably as strongly pronounced as in the adult human male.

(6). The axial line of the thigh can only be brought into correspondence with the axial line of the trunk by bending the trunk in the lumbar region, and thus increasing the lumbar curve until it attains a degree much in advance of that in man.

Orang-utan.—One specimen of the orang-utan I froze and cut in the mesial plane, under the same conditions as the three chimpanzees. It was a young female, which I purchased on the day of its death from Mr. Cross of Liverpool. It possessed its full complement of milk teeth, and measured 28 inches from the summit of the head to the sole of the foot when the lower limb was extended, so as to lie in the same line with the trunk. Bischoff[1] reckoned the age of a young orang, which measured 60 cm. ($23\frac{9}{10}$ inches), from the crown of the head to the heel, to be about four years. The specimen under consideration would, according to this computation, be at least five and a-half years old. I am very doubtful, however, if it had attained that age.

A very successful section was obtained of this animal, and a reduced representation of a tracing which was taken from the surface of the section while it was still in the frozen state is given in Plate III.

The statement which is made by Owen that in the orang there is but one curve, concave to the front, from the atlas to the sacrum, is seen to be erroneous. Nor, as Huxley has asserted, can the vertebral curvature in the orang, with truth, be compared with the curvature which is exhibited in a new-born child. It is likewise very different from what we find in the adult human spine, and in the chimpanzee. A glance at the tracing, however, will be sufficient to show that it bears a striking resemblance to the condition which is represented by Symington as present in a boy of six years old (Plate II.). There is the same sudden bend backwards in the upper dorsal region, and the same almost straight condition of the lower dorsal and lumbar regions. In the orang there is a feeble promontory and a sacro-vertebral angle of 158°; in the boy the promontory is also very deficient, and the sacro-vertebral angle 150°. And now, having noted these general points of resemblance, let us look more closely at the spinal curvature as it is exhibited in the orang figured in Plate III.

The cervical convexity is well marked, more especially in its lower part, and the axial curve of this segment of the column, as in man and

[1] "Ueber das Gehirn eines Orang-Outan," von Herr Prof. v. Bischoff. Sitzung. der math. phys. Classe der K. Vayer. Akad. der Wissens. München, vom. 17. Juni 1876.

in the chimpanzee, passes into the dorsal concavity at the central point of the intervertebral disc, between the 2nd and 3rd dorsal vertebræ.

The dorsal curve may be said to be limited entirely to the upper part of the region, and it is of a very abrupt nature; from the 7th dorsal vertebra down to the cartilaginous disc, between the 1st and 2nd lumbar vertebræ, the column is virtually straight. It is true that a close examination shows that no two vertebræ can be said to lie accurately superimposed, the one above the other; the axis lines of all the vertebræ, if prolonged, cut each other, and the 7th and 8th dorsal vertebræ stand a very small degree in front of those above and below; but to all intents and purposes we may regard this segment of the column as straight.

The lumbar curve is reduced to a minimum; but still it exists, and it is not so restricted as Broca and Topinard would have us believe. It involves the three lower lumbar vertebræ with the intervening cartilaginous discs. The convexity is very slight, and, curiously enough, the index of curve is identical with that of the spine of the boy figured by Symington (Plate II.), viz. 4·1. The special points, then, which are noticed in connexion with the vertebral column of the orang are:—

(1.) It resembles, in a striking manner, the spine of the boy, aged six, which has been figured by Symington.

(2.) The cervical curve presents the same limits as in man and the chimpanzee.

(3.) The segment of the column which succeeds this (viz., from disc between the 2nd and 3rd dorsal vertebræ to disc between the 1st and 2nd lumbar vertebræ) presents an upper curved portion, with the concavity to the front, and a lower straight part.

(4.) The lumbar curve is feeble, and involves the lower three lumbar vertebræ. It resembles that of man in being cut off from the sacrum, and differs from that of the chimpanzee in not including the first sacral vertebræ.

(5.) There is a feeble promontory.

In the orang the mobility of the lower limbs at the hip-joint is very

remarkable. It is a mistake to say that in man alone can the hip and knee-joints be fully extended, so that the leg is brought into a line with the thigh, and the thigh into a line with the trunk. The thigh in the orang can even be hyper-extended or dorsi-flexed, so as to form a very decided angle (153°), open to the back, with the axis line of the trunk, and yet produce no material change upon the condition of the vertebral column. The forcible bending back of the thigh beyond this point brings about a manifest increase of the lumbar curve; in other words, the further dorsi-flexion of the thigh can only be effected by a bending of the column in the lumbar region; and the further the limb is carried backwards the higher does the lumbar curve extend up the vertebral column. It is a question how far this extreme mobility of the lower limb at the hip-joint, as compared with the limited range of extension at the same joint in the chimpanzee, can be said to be associated with the feeble lumbar curve in the orang.

Gibbon.—The gibbon which I had an opportunity of examining was a nearly mature female specimen of the wauwau (Hylobates agilis), which I purchased from Mr. Cross of Liverpool. With its lower limb extended as far as it would go without using force, it measured thirty-one inches from the crown of the head to the sole of the foot. It was frozen and divided in the mesial plane in the same manner as the chimpanzees and the orang. The neck, unfortunately, was not accurately straightened before the freezing, and in consequence the saw passed through the cervical vertebræ a short distance to the left of the mesial plane, so as to cut the bodies close to their junction with the pedicles. The value of the section, however, was not much impaired, as the curves of the column could be accurately made out; and as the deviation was due merely to a twisting of the vertebræ, the head was cut as nearly as possible in the mesial plane.

It was with some interest that I examined the curves of this spine, seeing that I had been led to expect, from the description given by Broca, a lumbar curve more highly developed than in any other member of the anthropomorphic group. This is not the case, however: in this respect the gibbon stands intermediate between the chimpanzee and the orang; but with this difference, that the feeble curve of the orang, as well as the more

strongly marked curvature of the chimpanzee, appear to be more firmly stamped upon the vertebral column than in the case of the gibbon.

In Plate XII. a reduced tracing of the surface of the section of the gibbon is given. The spinal form presents some points of similarity to that of the boy (*æt.* 6) whose spine is figured in Plate II. The lumbar convexity, however, is more prominent, and in this respect it holds an intermediate position between the spine of the boy and that of the girl *æt.* 13, figured in the same Plate. The curve indices are as follows:—

Girl (*æt.* 13), ...	6·4
Gibbon, ...	5·0
Boy (*æt.* 6), ...	4·1

The lumbar curve involves the five lumbar vertebræ, and is cut off below from the slightly marked sacral concavity by a feeble promontory.

Such was the condition of the spinal curvature when the gibbon is placed upon its back; but when it was laid upon its side, after the section had been made and it had thawed, a complete change in the curvature took place. This was brought about by the ligamenta subflava, which are developed to an enormous extent, and more so in the lumbar than in the dorsal or cervical regions. These ligaments, by approximating the neural arches, caused the lumbar curve to extend up the column until it had involved the lower two-thirds of the dorsal region; so that a line drawn from the lower and anterior angle of the fifth lumbar vertebra to the upper and anterior angle of the fourth dorsal vertebra passed behind, and not in front of, the lower dorsal and upper lumbar regions (Fig. 8).

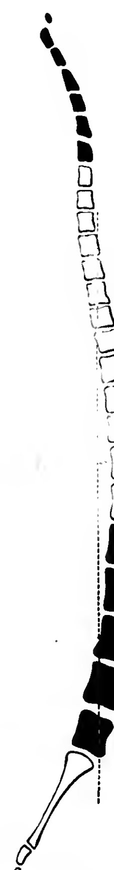


Fig. 8.

This extraordinary development of the ligamenta subflava in this gibbon (H. agilis) is clearly a provision for the purpose of increasing the elasticity and spring of the vertebral column, and must be of service in enabling it to make those remarkable flights from one tree to another for which it is so famous.

We have previously noted that Broca describes different degrees of lumbar curvature in different members of this group; and he gives to the saimang, in this respect, a pre-eminence over all the others. I am inclined, however, to believe that these apparent differences have been due to the agency of the ligamenta subflava, and to the position in which the animal has been examined.

The mobility of the thigh at the hip-joint is intermediate in degree between that of the chimpanzee and the orang. The lower limb can be extended until the axis line of the thigh forms, with the axis line of the trunk, an angle of 146°, open to the front. At this point the movement at the hip-joint ceases; and the further straightening of the limb can only be effected by bending the spine in the lumbar region, and thereby increasing the lumbar curvature.

It is a curious circumstance, that in the three anthropoid apes we have examined the degree of consolidated lumbar curve is in inverse proportion to the degree of mobility at the hip-joint: the more extended the range of movement at the hip-joint, the less prominent do we find the lumbar convexity.

Proportionate lengths of the different Segments of the Column in the Anthropoid Apes.—We have already called attention to some very interesting points in connexion with the relative length of each section of the vertebral column in man, and we have noticed characteristic peculiarities according to sex and age. It is well to compare these with measurements of the corresponding spinal regions in the chimpanzee and orang. Taking the anterior surface of the vertebral column as the standard, and equal to 100, we obtain for the different regions of the spine the following indices:—

	MAN.			ANTHROPOID APES.		
	6 Adult males.	5 Adult females.	3 New-born children.	2 Chimpanzees.	1 Orang.	1 Gibbon.
Cervical region,	21·8	21·6	25·1	23·9	25·9	20·5
Dorsal region,	46·5	45·8	48·5	45·6	49·5	47·7
Lumbar region,	31·7	32·8	26·4	30·6	24·6	31·8

In the case of the chimpanzee, the thirteenth dorsal vertebra was included in the lumbar segment, and measured with it; in the orang, the four lower true vertebræ were taken as representing the lumbar region; and in the gibbon the thirteen dorsal vertebræ were measured as the dorsal region.

We have already noted the great proportionate shortness of the lumbar region, and the corresponding proportionate length of the dorsal and cervical segments in the new-born child, as compared with the adult. The chimpanzee, in this respect, holds a somewhat intermediate position: its cervical region is relatively longer than that of the adult human subject, whilst its dorsal and lumbar regions are slightly shorter, but not in so marked a degree as in the new-born child. By referring to the tables which are given by Aeby,[1] in his able paper upon the "Age-differences of the Vertebral Column," we find that the proportionate lengths of the three segments of the column in the chimpanzee closely correspond with those of a boy of two years old.[2] For the latter Aeby gives the following indices:—

Cervical region, ...	23·3
Dorsal region, ...	46·7
Lumbar region, ...	30·0

The orang, while it differs widely in the relative length of the three segments of the vertebral column from the adult man, corresponds closely in this respect with the new-born child. Although the lumbar region of the orang is composed of only four vertebræ, the relative length

[1] Die Altersverschiedenheiten der menschlichen Wirbelsaüle. Archiv für Anat. und Entwicklungsgeschichte. 1879.

[2] It must be borne in mind, however, that the average age of the two chimpanzees which were measured was probably not more than three years. Further, there is a difference in the indices of the two specimens corresponding to what we find, at corresponding periods of growth, in man, viz.:

	Chimpanzee.	
	Male (say 4 years old).	Female (say 2 years old).
Cervical region, . . .	23·2	24·0
Dorsal region, . . .	45·9	45·0
Lumbar region, . . .	30·9	30·3

of this segment is very similar to that of a newly-born human infant. The gibbon in this respect shows a close correspondence with the adult human subject; and, although an additional vertebra is included in the dorsal region, the proportionate length of these two regions in man and the gibbon is very similar, viz. :—

Dorsal region, man (adult), . . .	46·5
Dorsal region, gibbon (adult), . .	47·7

It would be outside the scope of this memoir to take part in the controversy which has arisen with reference to the homologies which exist between the vertebræ of spines belonging to different animals. Welcker[1] maintains that the vertebræ are not numerically homologous with each other, but that the separate segments of different columns—lumbar, dorsal, cervical, and so on—may be regarded as homologous, and as extending or diminishing the number of their elements, according to the needs of the animal. Dr. Emil Rosenberg[2] of Dorpat has advanced a different theory. He does not regard the absent lumbar vertebra of the orang as being suppressed, but as being merely shifted downwards, so as to constitute in this animal the first sacral vertebra. He advocates, therefore, a numerical homology of the vertebræ of the column. In this he is opposed by Holl,[3] who regards the first sacral vertebra as a fixed quantity, and therefore primary.

At first sight the correspondence in relative length between the lumbar region of the orang and that of the human child, and also between the dorsal region of the gibbon and that of man, comprising as each does a different number of elements, might be considered as giving some measure of support to the view propounded by Welcker; on the other hand we must not forget the enormous proportional length which is attained by the different segments of the column in certain of the lower apes, and in certain of the quadrupeds. It is obvious, therefore, that measurements of the different regions of the spine will throw little or no light upon this question.

[1] "Archiv für Anat. und Entwicklungsgeschichte," His und Braune. 1881.

[2] Upon the development of the Vertebral Column, and upon the Os centrale in the Human carpus. Morph. Jahr., vol. i. 1875.

[3] Sitzungsberichte der Akademie der Wissenschaften zu Wien, lxxxv. 1882.

Lower Apes.

In endeavouring to arrive at a proper conception of the nature of the curvature of the lumbar section of the vertebral column in the lower apes, I have followed the same method which I adopted in the case of the anthropoid apes. At the same time I am fully aware that in placing the lower apes in a recumbent attitude I have given them a posture which they seldom assume in life, and which therefore must be regarded as being somewhat unnatural. Almost all my attempts to freeze these animals in a sitting or oblique attitude failed; and, even in the few cases where the experiment was successful, it was apparent that the results could not be compared with those obtained for the anthropoid apes which had been frozen in the recumbent position. One advantage the recumbent attitude undoubtedly has, and it is this: it cannot be charged with the production of curves which do not naturally exist. It must exert, indeed, a directly opposite effect, and tend to flatten the dorsal concavity, and also the lumbar convexity, when such a curvature exists. When curves, therefore, are obtained in the dorso-lumbar region by this method of freezing, we may reasonably conclude that they are more or less inherent in the vertebral column. With respect to the cervical region, the curvature is entirely dependent upon the position of the head, as we have seen to be the case in a newly-born child.

Two methods which I adopted for the purpose of obtaining the curves of the spine in the sitting posture gave fairly good results. In one case I placed a small monkey (Cercopithecus ruber, No. 1, Plate v.) in a long narrow glass preparation jar, with the thighs flexed, and the weight of the body resting upon the ischial tuberosities. After freezing it for a few hours, so as to stiffen the superficial soft parts, I took it out of the freezing mixture, and proceeded to obliterate all the lateral curvatures of the spine, taking the utmost care not to interfere with the antero-posterior curves. When this was accomplished the animal was replaced in the jar and the freezing completed. In the second case, the ape (Cercocebus fuliginosus,

No. 1, Plate v.) was placed upon its side on a table, and put into an attitude as nearly as possible resembling the sitting posture. A plaster of Paris jacket, which surrounded it on all sides, except that which was in apposition with the surface of the table, was then applied. When the plaster was firm the whole preparation was placed in a zinc box, and the animal frozen. The gypsum covering, however, was found to interfere very much with the freezing of the specimen. Before using the saw the plaster was removed.

In all, I have frozen and divided in the mesial plane fifteen different species of the lower apes, and the reduced tracings of the greater number of these sections are reproduced in Plates iv., v., and vi. The majority of these specimens I purchased from Mr. Cross of Liverpool, but some of the rarer forms I received from the Zoological Gardens in London, through the kindness of my friend the Prosector, Mr. F. E. Beddard. I have also to thank the authorities of Trinity College for their generosity in procuring for me such apes as died in the Zoological Gardens in Dublin.

That the lumbar convexity exists in a more or less permanent form in certain of the lower apes there cannot be a doubt. Thus we find it well marked in specimens of Macacus, Cercocebus, Semnopithecus, and of the New World monkey, Cebus capucinus, which have been frozen in the recumbent position with the thighs flexed upon the abdomen. The flexibility of the spine in these forms is very great; in life they constantly assume many postures which must obliterate completely the special curvature in the lumbar region. This cannot be regarded, however, as militating against the assertion that they possess such a curve. Even in the European, where the lumbar curve may be regarded as being more firmly established than in any other race of man, the convexity in the region of the loins is in a great measure, if not completely, effaced when the spine is flexed. It would be wrong, on the other hand, to assert that the lumbar curve in the lower apes is stamped so permanently on the spine as in man or the anthropoid apes.

In my examination into this matter I very early perceived that the position of the lower limb exerted a most important influence upon the condition of the lumbar region in all the lower apes. The moment a

monkey endeavours to assume an upright posture, by bringing the trunk more into a line with the thigh, a manifest increase of the lumbar convexity occurs. In other words, the power of extending the thigh at the hip-joint is limited, and the animal raises its trunk by bending the spine in the region of the loins.

Macacus.—I have examined three specimens of the macaque monkey, viz. two of Macacus nemestrinus, and one of Macacus rhesus, and tracings of the sections of these are given in Plate IV. Unfortunately, both examples of Macacus nemestrinus shifted in the freezing-box, giving rise to a lateral bending of the neck. It was found impossible, therefore, to obtain a section of the cervical region. The sections of the dorsal and lumbar segments, however, were altogether satisfactory.

Macacus nemestrinus No. 1 was frozen on its back, with its thighs flexed upon the abdomen. A very distinct sigmoid dorso-lumbar curve exists. The lumbar convexity involves the lower three lumbar vertebræ, and is more pronounced than in the orang. Two points in this section cannot fail to attract notice, viz.—(1) the condition of the last lumbar vertebra, and (2) the sacral obliquity. The fact that the last lumbar vertebra is deeper in front than behind, and thus presents an index considerably below the standard, is apparent to the eye. This is a character which distinguishes many of the macaque monkeys from the higher apes, and, as we have already seen, it constitutes a point of resemblance to man (*vide* p. 7). That it is not a constant feature, however, will be seen by referring to Table Q, p. 21. The sacral obliquity is also very apparent. St. George Mivart's[1] statement that "in most Primates the sacrum and lumbar vertebræ appear (as far as one can judge from skeletons alone) to lie almost, or quite, in one line, so that the promontory is very slightly marked," can hardly be regarded as true for any member of this order.

Macacus nemestrinus No. 2 was a much larger specimen, which was frozen on its back; but in this case the thighs were extended before the freezing was commenced. The effect upon the lumbar curve is

[1] "Axial Skeleton in the Primates."—*Proc. Zool. Soc.*, 1865, xxxvi.

very marked. It is distinctly increased both in degree and in extent. It now involves the five lower lumbar vertebræ, and is as pronounced as we find it in the chimpanzee.

Macacus rhesus gave very remarkable results. It was frozen on its back with the thighs flexed on the abdomen. In none of the lower apes is the lumbar convexity so strongly marked, and in none, with the exception of Cynocephalus, is the sacral obliquity so abrupt.

The lumbar curve, as will be seen in the tracing, involves the five lower true vertebræ, and is very nearly as prominent as in the chimpanzee. The sacrum forms a distinct promontory, and presents a concave anterior surface.

Cercocebus fuliginosus.—Two specimens of this species were examined. The first (Plate v., No. 1) was placed on its side in an attitude resembling the sitting posture, and was then enveloped in a plaster of Paris jacket and frozen. As might be expected, the dorsal curve is more marked than in those varieties which were frozen on the back; but although the trunk was slightly bent forwards, a lumbar curve is manifest. It is of small extent, however, and only involves the lower two true vertebræ. Had the weight of the upper part of the trunk and the upper limbs been transmitted downwards through the column, it is probable that the convexity would have been of greater extent.

No. 2, Plate v., gives the tracing of the mesial section of another specimen of the same species, which was frozen on its back, with the thighs flexed on the abdomen. Next to Macacus rhesus it presents the most perfect lumbar convexity which I have observed amongst the lower forms of ape. The obliquity of the sacrum is also well marked, although it is not so pronounced as in the Macaque.

No. 3, Plate vi., is a reduced tracing which was taken from No. 2 after it had thawed, and with the lower limb extended to the extent indicated in the drawing. The increase of the lumbar convexity by this proceeding is well illustrated; and the increase is not only one of degree, but also of extent, because now the curve includes the lower six lumbar vertebræ.

Semnopithecus entellus (Plate IV.).—The lumbar convexity in this species appears to be limited to the last two lumbar vertebræ. The last lumbar vertebra is deeper behind than in front, and the sacrum presents a considerable amount of obliquity. The straightness of the entire spine is remarkable.

Cercopithecus.—In no member of this group have I found a strongly marked lumbar curve when the animal was frozen on the back. The Malbrouch monkey (Cercopithecus cynosurus) of West Africa was the only one in which it was very apparent. In this specimen the curve included the lower three lumbar vertebræ.

In Cercopithecus mona, No. 1 (Plate V.), Cercopithecus campbelli, No. 1 (Plate IV.), and in Chlorocebus sabaeus, No. 1 (Plate VI.), all of which were frozen in the recumbent position, hardly a trace of the lumbar convexity can be detected. No. 2 tracing in each case represents the same animal after it had thawed, with the lower limb extended. By this proceeding a strong lumbar curve at once sprang into existence. In other words, the thigh could only be approximated to the line of the trunk by a bending of the spine in the lumbar region.

In Plate V. two tracings of Cercopithecus ruber are represented. No. 1 tracing was obtained from an individual of this species, which was frozen in a sitting posture in a tall glass jar. The curves of the column resemble those of man and the chimpanzee much more than those of the specimen of Cercocebus fuliginosus, No. 1, which was also frozen in a sitting attitude. In the Cercopithecus, however, the weight of the upper part of the trunk was transmitted downwards through the column. No. 2 tracing was taken from the same animal after it had thawed, and with the lower limb extended. The increase of the lumbar curve is manifest.

Colobus vellerosus.—Not a trace of the lumbar curve can be detected in Colobus. The dorsal and lumbar vertebræ form one continuous concavity, which is only interrupted below by the promontory (No. 1, Plate IV.). The wedge-shaped appearance of the last lumbar vertebra

and the obliquity of the sacrum are very marked, which shows that these conditions can exist independently of a lumbar convexity. It is possible, indeed, that a high degree of sacral obliquity may in some cases replace the lumbar convexity.

No. 2 gives a view of the changes which are brought about by extending the thigh. Even now the lumbar curve is not so pronounced as we might have anticipated. The bending appears chiefly to have taken place at the sacro-vertebral articulation. This is evident from the fact that the dorsal tilting of the pelvis has become much more apparent.

Cynocephalus anubis.—The baboon which I had the opportunity of examining was a very large specimen. It measured 27½ inches from the crown of the head to the root of the tail, and it was frozen like the others, on its back, and with its thighs flexed. A lumbar convexity can hardly be said to exist (Plate vi.); but still the lowest three lumbar vertebræ are not included in the dorsal concavity, but are placed as nearly as possible in a straight line. The dorsal tilting of the sacrum is very pronounced—more so, indeed, than in any other ape. St. George Mivart, in his important Paper[1] on the Axial Skeleton of the Primates, mentions this peculiarity. It is possible that this very character may, in a measure, have been the cause of the obliteration of a lumbar curve, because the greater the backward sweep of the pelvis, the greater will be the tendency for a lumbar convexity to become flattened, when a heavy animal is placed upon its back. On the other hand, it would almost appear as if the absence of a lumbar curve was compensated for, in many cases, by a more striking sacral obliquity.

New World Apes.—Two specimens of the Capuchin monkey gave evidence of a lumbar curve. The tracing of one of these is reproduced in Plate vi. (Cebus capucinus), and in this the convexity is very strongly marked, and involves the lower four lumbar vertebræ. The second specimen of this species, which was frozen under similar circumstances,

[1] "Proc. Zool. Soc., London," 1865.

and of which I have the tracing in my possession, did not exhibit so strong a lumbar convexity; but, on the other hand, the sacrum was placed more obliquely.

A Marmoset, which I received from the London Zoological Gardens, was unfortunately so much disturbed in the freezing box, that I only obtained a section of the lumbar region. This, however, gave clear indications of being the seat of a slight anterior convexity. The extent of this, or indeed its degree, it was not possible to estimate accurately, seeing that the section through the bodies of the vertebræ was very oblique.

Different appearance presented by the Vertebræ of different Apes when seen in section.—A glance at the plates in which the sections of the spines of the apes are represented will be sufficient to show that the bodies of the lumbar vertebræ present very different appearances in different apes. In Cynocephalus the bodies of the lumbar vertebræ approach in shape those of man and the anthropoid apes: the antero-posterior diameter is very nearly equal to the vertical diameter. The lumbar vertebræ of the macaque, in a less degree, are fashioned in a similar style; but in Cercopithecus the difference is very apparent. In this genus the length of the vertical diameter of the vertebral body is double that of the antero-posterior diameter. Further, the eye of itself is sufficient to determine that in this respect Colobus occupies an intermediate position between the macaques and Cercopithecus. To bring these facts out more clearly, and at the same time more accurately, I have measured the third lumbar vertebra in four races of man, and two children, the second lumbar vertebra in Troglodytes, and the fourth lumbar vertebra in several of the lower apes; and by taking the sagittal diameter of the centrum as equal to 100, we can frame an index which will express the relation between the vertical and sagittal diameters. According to this method of comparison, the higher the index is, the longer is the vertebral body in relation to its sagittal axis.

The following are the results; but it must be borne in mind that only one specimen of each was measured:—

RELATION OF VERTICAL TO SAGITTAL DIAMETER OF THE CENTRUM OF A LUMBAR VERTEBRA, TAKEN FROM THE MIDDLE OF THE SERIES.

Sagittal diameter = 100.

Negro,	75·0
Malay,	80·9
Chimpanzee,	82·1
Gorilla,	83·0
European,	88·2
Andaman,	90·7
New-born child,	93·0
Child, *æt.* 4,	100·0
Cynocephalus anubis,	102·1
Macacus nemestrinus,	123·0
Colobus vellerosus,	140·0
Semnopithecus entellus,	200·0
Cercopithecus,	200·0

Certain interesting points are brought out by the above indices. It is curious to find the Negro and Malay separated from the European and Andaman by Troglodytes. The small vertical depth of the bodies of the lumbar vertebræ in the Negro is peculiar, and is characteristic of the two Negro skeletons in my possession. It is further strange, considering the affinities which are said to exist between the Andaman and the Negro, that in this respect the former deviates so much from the latter. It would appear, however, that this difference is merely one of those infantile characters which Prof. Flower[1] has shown distinguish in so prominent a degree the Andaman race. By referring to the above Table

[1] "Journal of the Anthropological Institute." 1879.

it will be seen that the lumbar vertebra of a child aged four (or thereabouts) possesses an index of 100; in other words, the sagittal and vertical diameters are equal in length, and in this it closely approaches the baboon. In the new-born child the same deviation from the adult form is to be noted, but not in so marked a degree.

A very instructive index may also be formulated, by means of which the coronal or transverse diameter can be compared with the vertical diameter. We take the former as the standard, and as equivalent to 100:—

RELATION OF CORONAL DIAMETER TO VERTICAL DIAMETER OF THE CENTRUM OF A LUMBAR VERTEBRA, TAKEN FROM THE MIDDLE OF THE SERIES.

Coronal diameter = 100.

Gorilla,	60·4
Chimpanzee,	62·3
Negro,	62·5
Malay,	63·7
European,	65·4
New-born child,	67·0
Andaman,	72·4
Child, *æt.* 4,	77·0
Cynocephalus anubis,	92·0

In this case the gorilla and the chimpanzee head the list, and therefore show a greater proportionate breadth of the vertebræ than the others. The indices of the Negro, Malay, and European, however, are not very different, whilst the Andaman exhibits a decided approach towards the child and the baboon.

QUADRUPEDS.

On submitting the foregoing results to Professor Flower, of the British Museum, he was good enough to point out that the investigation would be incomplete unless the condition of the lumbar segment of the spine in a quadruped was examined. In adopting his suggestion, I encountered the same difficulty which had occurred to me in the case of the lower apes, viz., the position in which the animals should be frozen. If it is unnatural to place an ape upon its back, it is much more so to deal similarly with a dog. And yet it was necessary to do this in order that the results in each case might be compared with each other. That this experiment might be checked, I froze a second dog in a position as nearly approaching the standing attitude as it was possible so to do.

With the exception of Dr. Charpy, no one, so far as I know, has ever even hinted at the possibility of a quadruped possessing a lumbar curve. This author says[1]:—"I have been able to verify in a fresh squirrel, an animal with an intermittent erect attitude, at least for the trunk, that its long, supple lumbar region, which can be curved without difficulty into the arc of a circle, had an anterior convexity recalling, in every respect, that of the new-born child; the arch, occupying all the lumbar region, had its summit or maximum very low, at the disc between the sixth and seventh vertebræ." Whilst I cannot accept the accuracy of his statement with regard to the new-born child, I have every belief in the accuracy of his observation regarding the lumbar column in the squirrel.

The general view which is entertained by anatomists, and one which is figured in various treatises, may best be understood by quoting from the recent anatomical lectures by Dr. Adolf Pansch.[2] He says:—"It is of interest to take a glance at the form of the vertebral column in animals. The quadrupeds have essentially only two curves—a dorsal and a cervical. If

[1] "De la courbure lombaire," &c.—*Journ. de l'Anat. et de la Phys.* Paris, 1885.

[2] "Anatomische Vorlesungen." Berlin, 1884.

you take the caudal curve, then we obtain a serpentine vertebral column; but the relationships are quite different from man, because the spine is horizontal. The most important curve is that of the back; it carries, like a spring, the weight of the trunk and viscera, and this load draws downwards the arch, and acts, therefore, as a diminisher of the curve, whilst both ends of the curve are placed upon the support of the legs. The cervical curve arises through a pressure of the head operating in a horizontal direction, whilst the ligamentum nuchæ and the neck muscles from the front part of the dorsal curve hinder the head from falling downwards. These curves of quadrupeds have arisen first in the course of development, and have become permanent; they likewise suffer diminution and augmentation through increase and decrease of the weight. But what happens if these animals stand upright? If you observe a horse in the circus, with straightened hind-legs, or in a sitting posture, or if you study an upright-walking dog, you find the same neck and back curve as in quadrupedal walking—you find, in spite of the 'walking upon two legs,' *no loin bending in these animals.* The lumbar convexity is absent."

In Plate VI.*a* the reduced tracings of sections of the spine in three dogs, which were frozen in different attitudes, are figured, and in all an approximation to a lumbar curve is more or less manifest. No. 1 (a fox-terrier) was frozen on its back, with its limbs unrestrained. No. 2 (a large Scotch collie) was placed in a position as nearly resembling the standing attitude as possible. Having amputated the limbs just beyond the lower surface of the body the stumps were arranged so as to support the weight of the trunk, whilst the head was supported by means of a block placed under the lower jaw. No. 3 (a Maltese poodle) was frozen on its back, but the hind limbs were drawn backwards into the position which a dog frequently assumes when stretching himself.

In each region the curvature is very characteristic. The cervical region in both No. 1 and No. 2 shows a double curve, viz., a gentle concavity[1] downwards in its upper part, and an abrupt bend, convex downwards,

[1] The concavity in the cervical region is very probably a *post mortem* condition, and due to the relaxation of the muscles which give support to the head.

which also involves one of the dorsal vertebræ, where it passes into the dorsal concavity. The straightness of the cervical vertebræ in No. 1 is clearly due to the position of the head.

The dorsal curve begins at the second dorsal vertebra, and ends at the third lumbar vertebra, and it constitutes an arch which has its summit formed by the last dorsal vertebra.

The posterior lumbar vertebræ show a slight convexity downwards in No. 1 and No. 2, whilst in No. 3, where the lower limb was stretched, the lumbar curve is very strongly marked.

It is very evident, then, that the dorso-lumbar portion of a quadruped's spine does not form a simple arch, but that even in these animals there is exhibited a tendency to the formation of a lumbar convexity.

In Plate VI.*a* an outline tracing is given of the anterior curvature of a bear. This was obtained by moulding a strip of lead upon the front surface of the vertebral bodies, after the animal had been eviscerated, and while it was lying upon its back. It shows a very distinct lumbar convexity.

It has been too much the habit of anatomists to draw loose comparisons, upon insufficient data, between the infantile vertebral column and the spine of the apes. To this charge I must also plead guilty, in a lecture published in Edinburgh in 1881, under the auspices of the Health Society. Dr. Charpy,[1] reasoning in the same direction, arrives at conclusions which are different from those ordinarily in vogue, but which are nevertheless equally untenable. He says: "The human spine passes through a series of progressive phases, which correspond to animal forms: the fœtal column is that of a quadruped; the infantile column that of an anthropoid; the human type is the termination of a design roughly sketched in the animal or embryo."

At no stage in its development can the human column be regarded as corresponding in the character of its curvatures with that of the full-grown quadruped. The infantile column may be compared, however, in this

[1] "Journal de l'Anatomie et de la Physiologie." 1885.

respect, with the spine of Colobus or certain forms of Cercopithecus; although even in this case the comparison is somewhat strained. The orang, in its spinal form, closely approaches a boy aged six; the gibbon presents some features in the outline of its spinal curvature which are intermediate between those of a boy aged six, and a girl aged thirteen, but it is sharply marked off from the latter by its feeble promontory; whilst the chimpanzee exhibits a spinal curvature somewhat similar to that of an adult man. These comparisons, however, will not bear too close examination; and, although a general similitude may be noted, we may positively affirm that at no stage in its growth is the curvature of the human column an exact counterpart of that of the column of any of the lower animals in their full-grown condition.

BIBLIOGRAPHY FOR PART I.

AEBY: . . . . 1.—"Die Altersverschiedenheiten der menschlichen Wirbelsäule," Archiv für Anatomie und Entwicklungsgeschichte (His und Braune). 1879.

,, ... 2.—"Beiträge zur Osteologie des Gorilla."—Morph. Jahr., vol. iv.

,, ... 3.—"Der Bau des menschlichen Körpers."—Leipzig. 1871.

BALANDIN: . . . 4.—"Beiträge über die Entstehung der physiologischen Krümmung der Wirbelsäule beim Menschen."—Virchow's Archiv für Path. Anat. und Phys. 1873.

BARWELL: . . . 5.—"The Causes and Treatment of Lateral Curvature of the Spine." 1868.

BISCHOFF: . . . 6.—"Ueber das Gehirn eines Orang-outan." Sitzung der math.-phys. Classe Akad. München vom 17. Juni, 1876.

BOULAND: . . . 7.—"Rech. Anat. sur les courbures du Rachis chez l'homme et chez les animaux."—*Journ. de l'Anat. et de la Phys.*—Paris, 1872.

BRAUNE: . . . 8.—"Atlas of Topographical Anatomy" (translated by Bellamy). London, 1877.

BROCA: . . . . 9.—"L'ordre des Primates."—*Mémoires d'Anthropologie*, vol. iii. Paris, 1877.

CHARPY: . . . 10.—"De la courbure lombaire et de l'inclinaison du bassin."—*Journal de l'Anatomie et de la Physiologie* (Robin et Pouchet). Paris, 1885. No. 4.

CLELAND: . . . 11.—"Tracings of a Mesial Section of a Newly-born Child."—*Proc. British Association for Advancement of Science.* 1863.

CRUVEILHIER: . . 12.—"Traité d'Anatomie déscriptive." Paris, 1875.

DUCHENNE: . . . 13.—"Physiologie des Mouvements," &c. Paris, 1867.

,, . . . 14.—"Courbure lombo-sacrée"—Archiv de Méd. 1866.

FLOWER: . . . 15.—"Catalogue of the Osteological Specimens" in the Museum of the Royal College of Surgeons of England.

Part I.—Man. 1879.

Part II.—Mammalia. 1884. (Assisted by Dr. J. G. Garson.)

,, ... 16.—"Osteological Characters of the Andaman Islanders."—*Journal of the Anthropological Institute.* 1879.

FOSTER AND BALFOUR: 17.—"Elements of Embryology" (2nd. edit. by Sedgwick, 1885).

GEGENBAUR: . . 18.—"Lehrbuch der Anatomie des Menschen." Leipzig, 1883.

GOODSIR: . . 19.—"Anatomical Memoirs," edited by Sir William Turner, Edinburgh, 1868.

GUERLAIN: . . 20.—"Incurvation lombo-sacrée."—*Bulletin Soc. Anthrop.* 1867.

HARTMAN: . . 21.—"The Anthropoid Apes." London, 1885.

HAUGHTON: . . 22.—"Animal Mechanics." London, 1873.

HENLE: . . 23.—"Handbuch der Anatomie des Menschen." Erster Band.—Erste Abtheilung (Knochenlehre). 1871.

HIRSCHFELDT: . 24.—"Nouvel aperçu sur les conditions anatomiques des Courbures de la colonne vertébrale chez l'homme." *Gaz. Méd. de Paris.* 1849.

HIS: . . . 25.—"Anatomie menschlicher Embryonen." 1884.

HOLL: . . 26.—"Ueber die richtige Deutung der Querfortsätze der Lendenwirbel und die Entwicklung der Wirbelsäule des Menschen. Sitzungsberichte der Akademie der Wissenschaften zu Wien." lxxxv. 1882.

HORNER: . . 27.—"Ueber die normale Krümmung der Wirbelsäule."—Müller's Archiv. 1854.

HUMPHRY: . . 28.—"The Human Skeleton." Cambridge, 1858.

HUXLEY: . . 29.—"Man's Place in Nature." London, 1863.

,, . . . 30.—"The Anatomy of Vertebrated Animals." London, 1882.

HYRTL: . . 31.—"Lehrbuch der Anatomie." Wien. 1884.

KRAUSE: . . 32.—"Handbuch der menschlichen Anatomie."

LAGNEAU: . . 33.—"D'Anthropologie de la France." Bulletins de la Soc. d'Anthropologie de Paris. 1867.

LANGER: . . 34.—"Wachsthum des menschlichen Skeletes—Denkschriften der Kaiserlichen Akademie der Wissenschaften, Wien" (Math. Natur. Classe), Band XXXI. 1872.

LUSCHKA: . . 35.—"Die Anatomie des Menschen."

MACALISTER: . . 36.—"On the Anthropoid Apes": Lecture delivered before the Royal Dublin Society, Dec. 16, 1872.

,, . . 37.—"The Aesthetical Relations of the Curves of the Human Spine": read before the Royal Dublin Society, Feb. 17, 1870.

Merkel: . . . 38.—" Ueber den Bau der Lendenwirbelsäule :" Archiv für Anat. und Phys. (His und Braune), 1877.

Meyer: . . . 39.—" Das aufrechte Stehen." Müller's Archiv. 1853.

,, ... 40.—" Das aufrechte Gehen." Müller's Archiv. 1853.

,, ... 41.—" Die Individualitäten des aufrechten Ganges." Müller's Archiv. 1853.

,, ... 42.—" Lehrbuch der physiologischen Anatomie des Menschen." 1861.

,, ... 43.—" Die Statik und Mechanik des menschlichen Knochengerüstes." Leipzig. 1873.

Mivart: . . . 44.—" The Axial Skeleton in the Primates." *Proc. Zool. Soc.* 1865. No. xxxvi.

,, . . . 45.—" Man and the Apes." London. 1873.

Nühn: . . . 46.—" Untersuchungen und Beobachtungen aus dem Gebiete der Anatomie," &c. Erstes Heft. 1849.

Owen: . . . 47.—" Comparative Anatomy and Physiology of the Vertebrata." London. 1866.

,, ... 48.—" On the Osteology of the Chimpanzee and Orang-utan." *Trans. Zool. Soc.* 1835.

,, ... 49.—" Comparison of the lower jaw and vertebral column of the Troglodytes niger, Troglodytes gorilla, Pithecus satyrus, and different varieties of the Human Race." *Trans. Zool. Soc.*, Vol. iv., 1851.

Pansch: . . . 50.—" Anatomische Vorlesungen." Berlin. 1884.

Parow: . . . 51.—" Studien über die physikalischen Bedingungen der aufrechten Stellung und der normalen Krümmungen der Wirbelsäule." Archiv für pathologische Anat. und Phys. 1864.

Pirogoff: . . 52.—" Anatome topogr. sectionibus per corpus hum. congelatum triplici direct. ductis illustr. Petropol." 1859.

Pruner-Bey: . . 53.—" Mémoire sur les Nègres lu à la Société d'Anthrop." 1861.

Ravenel: . . 54.—" Die Maasverhältnisse der Wirbelsäule und des Rückenmarkes beim Menschen."—Zeitschrift für Anatomie und Entwicklungsgeschichte. 1877.

Regalia: . . 55.—" Casi di anomalie numeriche delle vertebre nell' uomo," &c. Estratto dall' Archivio per l'Anthropologia e l'Etnologia. 1880. Vol. x.

ROSENBERG: . . 56.—"Ueber die Entwicklung der Wirbelsäule und das Centrale Carpi des Menschen." Morphologisches Jahrbuch. Band I.

SAPPEY: . . 57.—"Anatomie Descriptive." 1876. Tome I.

SYMINGTON: . . 58.—"Curvature of the Spine in Children": Paper read in the Biological Section of the British Association for the Advancement of Science, in Aberdeen. 1885.

TOPINARD: . . 59.—"Eleménts d'Anthropologie générale." Paris. 1885.

,, . . . 60.—"Anthropology" (Library of Contemporary Science). Lond. 1878.

,, . . . 61.—"Des anomalies de nombre de la colonne vertébrale." Revue d'Anthrop. 1877.

TURNER: . . 62.—"Introduction to Human Anatomy." Vol. I.

VROLIK: . . 63.—"Tijdschr. voor de Natuurk. Welensch." Tome III. 1850.

WEBER: . . . 64.—"Mechanik der menschlichen Gehwerkzeuge von den Brüdern Wilhelm und Eduard Weber." 1836.

WELCKER: . . 65.—"Die neue anatomische Anstalt zu Halle durch einen Vortrag über Wirbelsäule und Becken eingeweiht von dem derzeitigen Director." Archiv. für Anatomie und Physiologie (His und Braune). 1881. Anatomische Abtheilung.

PART II.

Topographical Anatomy of the Chimpanzee, Orang, and Gibbon.

So far as I am aware, this is the first time that the method of dividing the frozen body in different planes has been employed for the elucidation of facts bearing upon Comparative Anatomy. Its range of utility is limited, but it is unquestionably a valuable means of enabling us to institute a closer comparison between the anthropoid apes and man. Certain relations of distinct morphological importance cannot be ascertained with accuracy by any other method; and further, by referring the position of the viscera to the different vertebræ, we can determine whether the assumption of the upright walk by man has led to any subsidence of the viscera.

Brain.—The question as to how far the cerebrum in the anthropoid apes projects backwards, in relation to the upper surface of the cerebellum, has given rise to much discussion. At the present moment we have convincing proof on the part of Gratiolet,[1] Huxley,[2] Flower,[3] Marshall,[4] Rolleston,[5] Turner,[6] and Wilder,[7] that, throughout the entire order of

[1] "Plis cerebraux de l'homme." 1861.

[2] "Man's Place in Nature," and also in *Proc. Zool. Soc.*, 1861, "On the Brain of Ateles."

[3] "On the Posterior Lobes of the Cerebrum of the Quadrumana." *Phil. Trans.* 1862.

[4] "On the Brain of a young Chimpanzee." *Nat. Hist. Review.* July, 1861.

[5] "On the Affinities of the Brain of the Orang." *Nat. Hist. Review.* April, 1861.

[6] "On the Anatomical Relations of the Surfaces of the Tentorium to the Cerebrum and Cerebellum in Man and the lower Mammals." *Proc. Roy. Soc.:* Edinb. 1862.

[7] At the meeting of the American Association for the Advancement of Science, in Philadelphia in 1884, Professor Burt G. Wilder, of Ithaca, N.Y., exhibited the brain of a young chimpanzee which he had hardened *in situ*. It was a very beautiful preparation, and showed a certain amount of projection of the posterior lobes beyond the cerebellum. (*Proc. American Association*, 1884, p. 527.)

Primates, with the single exception of the Lemurs, the posterior lobes of the cerebrum extend backwards beyond the cerebellum. Such being the case, it is not a little surprising to find an anatomist of the repute of Dr. H. C. Chapman,[1] in two of the more recent Papers which have appeared upon the structure of the anthropoid apes, expressing a doubt as to whether the posterior cerebral lobes in the chimpanzee can be considered as completely covering the cerebellum. In his latest article on the orang he says:—"The cerebellum of my orang was relatively larger than that of man, but smaller than that of either of the chimpanzees I have dissected, and was just covered, and no more, by the posterior lobes of the cerebellum." Further on he remarks: "In the account I gave of the female chimpanzee, I stated that I found the cerebellum uncovered. I had the opportunity a short time since of verifying that statement in the male, noticing *in situ* that the cerebellum was uncovered by the posterior lobes. This was found to be the case by Mr. Arthur Browne, the superintendent of the Phila. Zool. Gardens, in a third chimpanzee which died there. With all deference to Professor Marshall's photograph of a plaster cast of the brain of a chimpanzee, and however it may truthfully represent the relations of the cerebellum in his specimen, I must say that it would be simply monstrous if accepted as an illustration of either of mine, and with profound respect for Professor Huxley's opinion, regarding the interior of the skull being a guide for the determination of the proportion between posterior lobe and cerebellum, I find it anything but a safe one as regards the anthropoid apes. For the space between the posterior lobes of the brain and dura mater and bone, both posteriorly and laterally, I find very variable *in situ*, due to the state of the blood-vessels, and amount of fluid in arachnoid and subarachnoid cavities."

It may, perhaps, appear to be a work of supererogation on my part to reopen this question; but I think that it will be generally admitted that the precise extent of cerebral projection in the different members of the order

[1] "On the Structure of the Chimpanzee." *Proc. Acad. Nat. Sci. Philadelphia.* 1879. "On the Structure of the Orang." *Proc. Acad. Nat. Sci. Philadelphia.* 1880.

Primates cannot be regarded as having been absolutely defined; and where this has been attempted, certain inaccuracies, due to the method of investigation employed, have occurred.

It must be evident to everyone that there is only one way of obtaining absolutely positive and trustworthy information regarding the exact relationship which exists between the cerebrum and cerebellum, and that is by making sections through the frozen head. Whilst the parts are still in the frozen state, and therefore fixed immovably in their several places, tracings of the cut surface can be taken, and a permanent record of their mutual positions obtained. Undoubtedly, the next most reliable method consists in an inspection of the cranial cavity in section, or in the examination of a plaster cast of the interior of the skull; but it must be admitted that both of these plans are open to the objections which have been urged against them by Chapman. All dealings with the soft swaying brain, after the cranial wall has been removed, must be regarded as misleading; and the same also, but to a less extent, may be said for those methods, by which attempts are made to harden the brain *in situ*, by the injection through the carotids of preservative fluids. The best proof of what I say is to be found in the strange want of unanimity of opinion which exists upon a point apparently so simple as the relative extent to which the posterior lobes of the cerebrum project beyond the cerebellum.

With the object of determining the point at issue, I have made sections of the heads of two chimpanzees, one orang, one gibbon (Hylobates agilis), a certain number of the lower apes, one adult man, and two newly-born children. In each case a definite plan was followed. The first section was made as far as possible in the mesial plane, with the view of dividing the central lobe or vermiform process of the cerebellum. The second section was made in the sagittal direction, and parallel to the first, a short distance to the outer side of the mesial plane, so as to cut the posterior point of the cerebral hemisphere, and the most bulging part of the cerebellar hemisphere; lastly, a third sagittal section was made parallel to the others, and in such a plane that it passed through the middle of the eyeball. The drawings which are reproduced in

Plates X., XI., and XII., were constructed from accurate tracings, taken from the more important of these sections while still in the frozen state. The reduced tracing of the orang, which is given in Plate III., represents the mesial relationship of the cerebrum to the cerebellum. In all the sections, with the exception of those of the gibbon, the cerebrum was found to project backwards in a marked degree beyond the cerebellum.

The relative extent of this projection in the mesial plane in the different specimens is given in the following Table. All the measurements which are indicated in this and the succeeding Tables were made with the head in such a position that the face looked directly forwards. Professor Burt G. Wilder remarks, with truth, that the relative position of parts depends entirely "upon the way in which the brain is held."

DISTANCE WHICH THE CEREBRUM PROJECTS BEYOND THE CEREBELLUM IN THE MESIAL PLANE, *i. e.* WHERE THE SECTION HAS PASSED THROUGH THE VERMIFORM PROCESS, OR CENTRAL LOBE OF THE LATTER.

	Actual amount of projection in millimetres.	Proportion—taking Cerebrum as equal to 100.
Adult male,[1]	25·5 mm.	15·9
Adult female (Braune's Atlas), . . .	27·0 mm.	17·0
Newly-born male child,[1]	9·0 mm.	9·0
Newly-born female child,[1] . . .	7·0 mm.	7·5
Male chimpanzee,[1 2]	10·0 mm.	10·0
Orang,[1]	10·0 mm.	10·6
Baboon, ...	10·5 mm.	11·2
Cebus capucinus, ...	5·5 mm.	8·7

[1] All these specimens are mounted as permanent preparations in the Anatomical Department of Trinity College, Dublin.

[2] The measurement which is given here was not obtained from the tracing from which the large coloured Plate of the male chimpanzee was constructed, because this section did not pass accurately through the vermiform process.

The projection of the posterior lobes of the cerebrum beyond the cerebellum is most strongly marked in the mesial sections, because in this plane it is the vermiform process which is cut, and the antero-posterior length of this is considerably less than that of the cerebellar hemisphere. In the male chimpanzee and the orang the amount of projection measured 10 mm.; in the adult human male which I examined it amounted to 25·5 mm., whilst in the mesial section of a female represented by Braune[1] it measures 27 mm. If we regard the length of the cerebrum as being equal to 100, the proportion in the different specimens will be found to be expressed by the following numbers:—

Man (average of two specimens), . . .	16·4
Orang, ...	10·6
Chimpanzee, ...	10·0

In the newly-born child it is interesting to note that the distance to which the posterior lobes of the cerebrum are carried beyond the cerebellum is not nearly so great as in the adult, and indeed not so great as in the chimpanzee or orang. It corresponds more to what we find in the Cebus capucinus, and the proportion is—

Human child (newly-born), . . .	8·2
Cebus capucinus, ...	8·7

But the baboon (Cynocephalus anubis) approaches man in this respect more closely than either of the two anthropoid apes. The proportion in this animal was 11·2. Referring to the great backward projection of the cerebral lobes in this animal, Flower remarks:—"Gratiolet has demonstrated that the principal cerebral characteristic of this genus is the great development of the occipital lobes. In a nearly full-grown example of C. porcarius I find that they project $\frac{4}{10}$ths of an inch beyond the cerebellum, or rather more than $\frac{1}{9}$th of the entire length of the hemisphere, proportionally more, therefore, than in man." 1

[1] "Atlas of Topographical Anatomy."

can confirm the accuracy of this statement in all respects, except in so far as the comparison with man is concerned. Calculating the proportional projection of Professor Flower's specimen by the method employed above, we obtain an index of 11·1. This is almost identical with the index I have obtained for C. anubis; but it is very far below that of an adult man (16·4).

Let us now examine the results obtained by dividing the frozen head a very short distance to the outer side of the mesial plane. As a standard of comparison in man I have taken the drawing of a mesial section of a male adult by Professor Braune.[1] In this the section is represented as deviating slightly to one side of the mesial plane, but still it cannot be regarded as affording an altogether satisfactory standard of comparison. In Plate xi., Fig. 2, is a full-size drawing of a section of the head of a female chimpanzee nearly in the same plane, but slightly oblique, and a little further out; and in Plate xii., Fig. 2, a section of the head of the gibbon in a plane very similar to that of Braune's male is represented. In both man and chimpanzee the projection of cerebrum beyond cerebellum is not so marked at this point as in the mesial plane; but man still maintains a pronounced pre-eminence, as will be observed by comparing the indices given in the subjoined Table.

DISTANCE WHICH THE CEREBRUM PROJECTS BEYOND THE CEREBELLUM, AS SEEN IN A SAGITTAL SECTION OF THE HEAD IMMEDIATELY TO THE OUTER SIDE OF THE MESIAL PLANE.

	Actual amount of projection in millimetres.	Proportion—length of Cerebrum = 100.
Adult human male (Braune), . . .	23·5 mm.	14·5
Female chimpanzee[2] (Fig. 2, Plate xi.), .	8·5 mm.	8·8
Female gibbon[2] (Fig. 2, Plate xii.) . .	3·0 mm.	4·0

[1] "Atlas of Topographical Anatomy."

[2] These specimens are mounted as permanent preparations in the Anatomical Department of Trinity College, Dublin.

Perhaps the most instructive section of all is that which is taken in the line of the eyeball. By such a section one realizes more fully the different degrees of projection, and above all the distance which separates man in this respect from the other members of the order Primates. In Plate x. sections, in this plane, of the heads of the male chimpanzee (Fig. 1), Cebus capucinus (Fig. 2), and man (Fig. 3), are depicted; in Plate XI. a corresponding section of the head of the female orang is represented (Fig. 1); and in Plate XII. (Fig. 3), a similar section of the head of the gibbon is figured. We can readily detect by the eye the different degrees of projection in each case; but the precise measurements are given in the following Table :—

DISTANCE WHICH THE CEREBRUM PROJECTS BEYOND THE CEREBELLUM, AS SEEN IN A SAGITTAL SECTION THROUGH THE HEAD TAKEN SO AS TO BISECT THE EYE-BALL.[1]

	Actual amount of projection in millimetres.	Proportion—length of Cerebrum = 100.
Adult human male (Plate x., Fig. 3), .	25·0 mm.	15·4
Newly-born female child, . . .	11·0 mm.	11·5
Male chimpanzee (Plate x., Fig. 3), .	7·0 mm.	8·6[2]
Female chimpanzee, ...	6·0 mm.	6·3
Female orang (Plate XI., Fig. 1), . .	7·0 mm.	8·1
Female gibbon (Plate XII., Fig. 3), .	0·0 mm.	0·0
Cebus capucinus (Plate x., Fig. 2), .	5·5 mm.	9·6

The gibbon is remarkable for the small extent to which the posterior

[1] All these specimens, with the exception of Cebus, are preserved as permanent preparations in the Anatomical Department of Trinity College, Dublin.

[2] The proportion expressed here is, perhaps, slightly above what it should be, because the section having passed a short distance to the outer side of the middle of the eyeball, the full length of the cerebrum was not obtained.

lobes of the cerebrum project beyond the cerebellum. Immediately to the outer side of the mesial plane the projection in my specimen amounted to only 3 mm. (Plate XII., Fig. 2), whilst in the line of the eye-ball the posterior limits of cerebrum and cerebellum corresponded exactly; or, if there was any difference at all, it was in favour of the cerebellum (Plate XII., Fig. 3). Huxley has already called attention to this peculiarity in the gibbon.[1] The siamang, he states, "is remarkable for the short posterior lobes of the cerebrum, which in this anthropomorphous ape do not overlap the cerebellum as they do in all the others." On the other hand Bischoff,[2] in his memoir upon Hylobates leuciscus, asserts that in his specimen the cerebellum was completely covered by the occipital lobes; and further, he disputes the statement that in the gibbon there is any remarkable reduction of the cerebrum as compared with the other anthropoid apes.

In Plate XI., Fig. 3, the drawing of a coronal section of the head of the female chimpanzee is given. The section passed through the widest part of the cerebellum, and comparing it with a corresponding section which I have made through the adult human head, I find that proportionally the extent to which the cerebrum overlaps the cerebellum is the same in both.

It appears to me that a good deal of the diversity of opinion that has been expressed upon this subject has been due to a want of perception of the great extent to which the posterior cerebral lobes project beyond the cerebellum in man. Thus Dr. Chapman[3] remarks: "Why should it be necessary to replace the brain of the chimpanzee or the orang in the skull, to make plaster casts, &c., if there is no difference between their brains and those of man and the monkeys, for there is no necessity of having recourse to such measures to prove that the cerebellum is covered in the latter?" In the human brain, when removed from the cranial cavity, we find little or no projection of the posterior lobes beyond the cerebellum; and when I removed the right half of the brain of the orang which is

[1] "Anatomy of Vertebrated Animals," p. 411. 1882.
[2] "Beiträge zur Anatomie des Hylobates leuciscus." München, 1870.
[3] "Structure of the Orang-outang."—*Proc. Acad. Nat. Sci. Philadelphia.* 1880, p. 13.

figured in Plate III., I found that it required some manipulation to place the cerebrum in such a position that it would completely cover the cerebellum. In the one case the swaying forwards of the cerebrum upon the crura cerebri causes the posterior lobes to advance until they become flush with the posterior border of the cerebellum; in the other, by the same process, a portion of the cerebellum is left bare.

Prof. Huxley[1] claims for the Chrysothrix a pre-eminence over man in the extent to which the posterior lobes of the cerebellum are carried backwards, and he supports this claim upon the evidence afforded by an examination of the interior of the skull in each. Unfortunately I have not had an opportunity of making sections of the frozen head of this animal.

Prof. Flower[2] calls attention to the fact that "on comparing the form of the brain in Cercopithecus with that of the human subject very great similarity is seen in the contour of the posterior half of the cerebrum; but the anterior lobes in the monkey are much reduced, being narrowed almost to a point, flattened, and largely excavated in the orbital regions." This similarity of the posterior portion of the cerebrum in certain of the lower apes to that of man is brought out in a very striking manner by means of the sections of the frozen head, and the plane in which it is most apparent is in that in which the head is divided in the line of the eyeball. In Plate X., Fig. 2, which represents a section through the head of Cebus capucinus, and Fig. 3, a section in the same plane through the head of man, render the correspondence of outline of the back part of the brain in these forms very manifest, and it is evident that in this respect Cebus resembles man more than either the chimpanzee (Plate X., Fig. 1) or the orang (Plate XI., Fig. 2). At the same time it should be noted that Cebus presents a greater fulness, and a greater relative depth of the posterior lobes, than man.

It is true that in Cebus the difference between the anterior lobe of

[1] "Man's Place in Nature."—See page 79, where the interior of the skull of an Australian is figured alongside that of the Chrysothrix.

[2] "Phil. Trans.," 1862.

the cerebrum and that of man is very marked. Its narrow and pointed character, and its strong orbital excavation, as exhibited in Plate x., stand out in strong contrast to the full, massive, and rounded frontal lobe of man. At the same time it will be noticed that the frontal lobe apparently advances further forwards than in man. If we let fall a plumb-line from the fore extremity of the cerebrum in Cebus, it cuts the eyeball immediately behind the lens, whereas in man such a line falls behind the equator of the eyeball. Further, if a vertical line be drawn upwards from the tip of the temporo-sphenoidal lobe, and the distance in each be measured from this line to the anterior extremity of the frontal lobe, it will be found to be relatively greater in Cebus than in man, in the proportion of 23·6 to 18·2.

Unfortunately the same comparison cannot be instituted in the case of the male chimpanzee (Plate x., Fig. 1), because the section is slightly oblique, and in front it cuts the eyeball a little to the outer side of its middle point. In the female chimpanzee and in the orang (Plate xi., Fig. 1) accurate sections were obtained, and the line drawn vertically downwards from the extremity of the frontal lobe cuts the eyeball in both a short distance behind the equator, whilst in the gibbon (Plate xii., Fig. 3) a corresponding line cuts the eyeball at a point still further back. Of course we cannot place much confidence in this relation between the frontal lobes and the eyeball, because the position of the latter cannot be regarded as being a fixed standard for purposes of comparison[1]; still it is interesting to observe in the drawing of Cebus that if the anterior bounding-line of the frontal lobe, as it ascends from the Sylvian fissure, be carried upwards with a gentle curve, so as to cut off the pointed extremity of

[1] It should also be borne in mind, that a very slight deviation of the section to one side or the other will produce a marked difference in the relation of the anterior limit of the cerebrum to the eyeball; and it is difficult to get a series of sections in precisely the same line. Thus, Rüdinger ("Topographisch-chirurgische Anatomie des Menschen. Dritte Abtheilung") represents a sagittal section of the human head which in front passes through the eyeball, apparently a little to the inner side of its central point. In this drawing a vertical line drawn downwards from the anterior extremity of the cerebrum passes through the crystalline lens. The outline of the entire cerebrum is peculiar, and differs in a marked degree from Fig. 3, Plate x.

the frontal lobe, the entire brain will then present a contour which exhibits a considerable resemblance to that of the chimpanzee or orang.

In the section of the chimpanzee's head (Fig. 1), and in the section of the human head (Fig. 3), figured in Plate x, the posterior and descending cornua of the lateral ventricle of the right side are opened. If the axes of these two recesses be prolonged, they will be observed to cut each other at a very different angle in the two sections. In the chimpanzee the posterior and middle cornua lie very nearly in the same line, and their prolonged axes form an angle of about 160°; in man, on the other hand, they both slope more or less downwards, and their axes cut each other at an angle of about 140°. In the chimpanzee, as we have seen, the section is not quite in the same plane as in the human head, and therefore the hippocampus minor has escaped division, but it can be seen on the lower and inner walls of the posterior cornu.

In the three brains of the chimpanzee which I have had an opportunity of examining (two by frozen sections, and one after removal from the cranial cavity), I was much struck by the large size of the hippocampus minor, and, on comparing it with the corresponding structure in the European brain, it was observed to be relatively of larger size. Professor Flower[1] has noticed the same point. In his memoir upon the posterior cerebral lobes he remarks:—"The hippocampus minor is one of the most striking characteristics of the typical simian brain, as it is greatest in Cercopithecus, Macacus, Cynocephalus, and Cebus, less in the anthropoid apes, and least of all, in proportion to the mass of cerebral substance contained in the lobe, in man." Marshall,[2] in his able account of the bush-woman's brain, states that the posterior horn of the lateral ventricle is long in proportion to that of the European, and that the hippocampus minor was of large dimensions, and measured 1·1 inches in length.

Two years ago I received from the late Dr. Hart, of Sierra Leone, who contributed so many valuable specimens to the Trinity College Museum, and whose untimely death we have recently had cause to deplore, six negro brains. Of these, only two were hardened sufficiently well to

[1] "Phil. Trans.," 1862. [2] "Phil. Trans.," 1864.

admit of an examination of the internal parts, but in both the hippocampus minor was seen to be of larger dimensions than is usually the case in the European.

The measurement of the hippocampus minor is a difficult matter in the human brain, on account of its very gradual subsidence as we trace it backwards. It is therefore not easy to make out the exact point at which it ends. In three Europeans, two negroes, and three chimpanzees (taking the hippocampus major as the standard, and equal to 100), I found the approximate proportional length of hippocampus minor to be as follows:—

European, ...	72·3
Negro, ...	77·6
Chimpanzee, ...	84·4

It is right to add, that Tiedemann[1] asserts that the hippocampus minor of the negro is in no respect different from that of the European.

I am unable to include the orang in the above comparison, because in the left half of the brain, which was removed after the animal had been divided in the mesial plane, there was not a trace of the posterior horn of the lateral ventricle. The calcarine fissure was strongly marked and very deep, but owing to the absence of the ventricular cavity in the posterior lobe of the cerebrum there was no free hippocampus minor. Of course this must be looked upon as an individual peculiarity: the presence of the posterior cornu and the hippocampus minor in the orang has been thoroughly established by several observers. Still it is an interesting anomaly, and it suggests the possibility that Sir Richard Owen, in his statements regarding the posterior lobe, posterior ventricular horn, and hippocampus minor, may, in the first instance, have been misled by an abnormal brain of this kind.

[1] "On the Brain of the Negro compared with that of the European and the Orang-Utan."—*Phil. Trans.*, 1836.

In the gibbon, owing to the reduction of the posterior lobes, the hippocampus minor is very short. In breadth and prominence, however, it is considerably in advance of the chimpanzee, and it reaches backwards to within 5 mm. of the posterior extremity of the occipital lobe. Its proportional length to the hippocampus major is as 70·8 to 100.

Professor Flower, in his exhaustive memoir upon the brain of the Quadrumana,[1] suggests a very useful and reliable method of obtaining a proper conception of the relative length of the posterior lobes. He says:—"As it seemed desirable to possess an exact means of estimating the length of the posterior lobes in different animals by a criterion derived from internal structure, I have taken the most prominent part of the convex border of the hippocampus major as the limit between the antero-median and posterior portions of the cerebrum. In man and the Quadrumana the angle formed at the junction of the hippocampus major and minor readily indicates the exact spot on which to place the compasses." The relative length of these two divisions of the brain may be compared in different animals, by taking the antero-median portion as the standard and equal to 100. Professor Flower gives the index for man calculated in this manner as 53; and for the orang as 50. Professor Marshall, for the brain of the bushwoman, obtained an index of 54·6, indicating a greater proportional length of the posterior lobes than in the European; and for the chimpanzee an index of 52. In two well-preserved negro brains I obtained an index of 53, which is identical with that given by Professor Flower for the European; and in three chimpanzees I obtained an average index of 50·3. It should be noted, however, that this low index for the chimpanzee is due to an unusual shortness of the posterior lobes in the female specimen; the individual indices were 51·8, 52·3, and for the female, 47·0. My orang gave an index of 52·8; whilst the gibbon yielded an index of 41·1.

A Table, containing the results indicated above, may be constructed thus:—

[1] "Phil. Trans.," 1862.

LENGTH OF ANTERO-MEDIAN AND POSTERIOR PORTIONS OF THE BRAIN AS DEFINED BY PROFESSOR FLOWER.

			Actual length in millimetres.		Proportion.		Average proportion for the posterior portion.
			Antero-median portion.	Posterior portion.	Antero-median portion.	Posterior portion.	
MAN.		EUROPEAN, . . . (Prof. Flower).	—	—	100·0	53·0	53·0
MAN.	NEGRO.	Aku, ... from Sierra Leone.	105·0	57·0	100·0	54·2	53·0
MAN.	NEGRO.	Timanee, ... from Sierra Leone.	108·0	56·0	100·0	51·8	
MAN.		BUSHWOMAN, . . (Prof. Marshall).	—	—	100·0	54·6	54·6
ANTHROPOID APES.	ORANG.	No. 1. . . . . . (Prof. Flower).	—	—	100·0	50·0	51·4
ANTHROPOID APES.	ORANG.	No. 2, ... figured in Plate XI.	61·5	32·5	100·0	52·8	
ANTHROPOID APES.	CHIMPANZEE.	No. 1, ... (Prof. Marshall).	—	—	100·0	52·0	50·8
ANTHROPOID APES.	CHIMPANZEE.	No. 2 (Male). . .	59·0	31·0	100·0	52·3	
ANTHROPOID APES.	CHIMPANZEE.	No. 3 (Female). . .	68·0	32·0	100·0	47·0	
ANTHROPOID APES.	CHIMPANZEE.	No. 4, ... from Sierra Leone.	54·0	28·0	100·0	51·8	
ANTHROPOID APES.		FEMALE GIBBON, . .	53·5	22·0	100·0	41·1	41·1

The foregoing facts all go to sustain the statement which has been made by Huxley regarding the shortness of the posterior lobes in the gibbon. In fact, as will be seen in the above Table, they are at least ten

per cent. shorter than the corresponding lobes in the chimpanzee or orang, and thirteen per cent. shorter than in man.

One of the most important differences between the brain of man and that of the anthropoid apes is to be found in the condition of the corpus callosum. The very varying degrees of development which this structure exhibits throughout the mammalian class might indeed lead us to expect this. It is true that its function is buried in obscurity; and this is all the more the case since a new light has been shed by Prof. Hamilton[1] upon the connections established by its fibres. Certain it is, however, that the degree of development exhibited by the corpus callosum bears a very direct relation to the degree of intelligence exhibited by an animal. The brain of anthropoid apes, as compared with that of man, is distinguished by the diminutive size of the corpus callosum, and there are good grounds for the belief that appreciable differences in this respect may also be found to exist between the different races of man. Thus, Marshall in his description of the bushwoman's brain, says:—"compared with the area of the internal surface of one hemisphere, the sectional area of the corpus callosum is in the bushwoman's brain as 1 to 25, in the European as 1 to 12·5, and in the chimpanzee as 1 to 28·5; so that the corpus callosum, thus estimated in proportion to the cerebrum, is in the bushwoman, only half as large as in the European, and not much larger proportionally than in the chimpanzee."

Measurements of the corpus callosum can only be satisfactorily made in mesial sections of the frozen head. As soon as the brain is removed from the cranium, the corpus callosum loses its finely arched form, and its proportions are altered. In the two succeeding Tables I give the dimensions of the corpus callosum, ascertained upon the frozen heads of three adult human subjects—a girl of thirteen years old, a boy of six years old, and two newly-born children. The measurements of two of the adults were taken from Braune's[2] representations of a male and female in mesial section; those of the girl and boy were obtained from drawings of similar sections by Dr. Symington of Edinburgh; whilst the others were measured directly from sections made by myself.

[1] "Journn. Anat. and Phys.", 1885.

[2] "Atlas of Topographical Anatomy."

Taking the entire length of the cerebral hemisphere as 100, indices can be calculated which give us a ready means of comparing the length of the corpus callosum with the extent of cerebrum in front of it, and the extent of cerebrum behind it. In the following Table these are termed, respectively—callosal length, pre-callosal length, and post-callosal length.

LENGTH OF CEREBRUM = 100.

	Pre-callosal length.	Callosal length.	Post-callosal length.
Adult human female, (Braune).	18·6	47·5	33·9
Adult human male, (Braune).	22·3	44·7	32·9
Adult human male,	22·8	43·8	33·2
Girl *æt.* 13, (Symington).	23·8	42·5	33·7
Boy *æt.* 6, (Symington).	22·4	41·0	36·5
Newly-born female child,	29·0	35·5	35·5
Newly-born male child,	25·7	36·1	38·2

	Average Pre-callosal length.		Average Callosal length.		Average Post-callosal length.	
	Absolute length.	Percentage length.	Absolute length.	Percentage length.	Absolute length.	Percentage length.
Three adults,	34·4 mm.	21·2	73·3 mm.	45·3	54·0 mm.	33·3
Girl *æt.* 13, (Symington).	41·0 mm.	23·8	73·0 mm.	42·5	58·0 mm.	33·7
Boy *æt.* 6, (Symington).	35·0 mm.	22·4	64·0 mm.	41·0	57·0 mm.	36·5
Two newly-born children,	24·5 mm.	27·2	33·5 mm.	35·8	34·5 mm.	36·8

The corpus callosum in the newly-born child is very different from that of the adult. It is relatively much shorter; it stands to the latter as 35·8 to 45·3. On the other hand, both the post-callosal and pre-callosal lengths are relatively much greater. In the boy of six years old, the corpus callosum is still shorter than in the adult; but its growth has taken place chiefly in an anterior direction, so that whilst the pre-callosal length corresponds with that of the adult, the post-callosal length is relatively as long as in the new-born child. The girl of thirteen approaches in this respect the adult form. By this time, therefore, the corpus callosum may be considered to have very nearly attained its full development. In the young embryo the front extremity of the corpus callosum is first called into existence, and it gradually extends backwards from this. In the new-born child, therefore, one would naturally expect to find the deficiency chiefly behind; but this is not the case. The deficiency is as marked in front as it is behind, and in its further development it appears to extend in the first place forwards, and afterwards to extend backwards, until it gains the proportion which it exhibits in the adult.

I applied the same test to the six negro brains which I had received from Sierra Leone, but as these had all been hardened after their removal from the cranial cavity, we cannot place much reliance upon the results. The arched form of the corpus callosum was gone, and, in consequence, its average length in the six specimens was greater than in the Europeans. Still it was evident that the depth of the corpus callosum in the negro, especially at its two thickened extremities, was less than that of the European. The following are the average indices which were obtained:—

Six Negroes,	Pre-callosal length,	19·4
	Callosal length,	47·0
	Post-callosal length,	33·5

In the anthropoid apes the corpus callosum is of small size, as compared with the same structure in an adult man. Not only is it shorter,

but its vertical depth is distinctly less in proportion to the size of the brain. Further, in the orang, and also in the chimpanzee, the front extremity or genu, when seen in mesial section, presents a different appearance. The thickening of this portion is feebly marked, and the bend which it takes is not sharp and sudden, as in man, but gentle and gradual, so that it presents a broad and bold convexity to the front, and a wide concavity to the back. This is particularly well seen in the orang (Plate III.). In the orang, also, the corpus callosum is considerably shorter than in the chimpanzee. The following Table gives its relative length in three anthropoid apes, and in three of the lower apes.

	Pre-callosal length.		Callosal length.		Post-callosal length.	
	Absolute length.	Percentage length.	Absolute length.	Percentage length.	Absolute length.	Percentage length.
One chimpanzee, . .	29·0 mm.	27·1	39·0 mm.	36·4	39·0 mm.	36·4
One orang, . . .	25·0 mm.	26·5	29·0 mm.	31·0	40·0 mm.	42·5
One gibbon, . . .	19·0 mm.	25·3	32·0 mm.	42·7	24·0 mm.	32·0
One baboon, . . .	20·5 mm.	21·8	41·0 mm.	43·3	33·0 mm.	34·9
One colobus, . .	13·0 mm.	20·3	29·0 mm.	41·6	26·5 mm.	38·1
One cebus, . . .	16·0 mm.	25·0	27·0 mm.	42·0	21·0 mm.	33·0

The close resemblance which this Table brings out between the chimpanzee and the newly-born child is remarkable. The orang differs in presenting a relatively shorter corpus callosum, and a greater post-callosal length. Indeed, in no form which I have examined is the post-callosal length so great. In the gibbon the callosal length is relatively much greater than in the other anthropoid apes, and in this respect it makes a decided approach towards man, and, at the same time, resembles closely

certain of the lower apes. On the other hand, the post-callosal length is shorter than in any ape examined.

Another curious point is indicated in the above Table, viz., that the corpus callosum of the baboon presents very nearly the same relative length as in man. To a certain extent this might have been anticipated, seeing that the backward projection of the posterior lobes of the cerebrum in this animal is more marked than in any other form we have examined. Not only is the callosal length similar, however, but so also is the post-callosal length and the pre-callosal length.

Both Cebus and Colobus are, likewise, less removed from man in this respect than are the anthropoid apes. Cebus differs chiefly in presenting a slightly shorter corpus callosum and a longer pre-callosal region. The greater length of the latter has already been pointed out, and it is even evident to the eye in Figure 2, Plate x. Colobus differs in a marked degree from Cebus, in its great post-callosal length. In this respect it resembles the orang, but deviates from it in the greater relative length of the corpus callosum.

Spinal cord.—In the different anthropoid apes the distance which the spinal cord extends downwards in the spinal canal is by no means the same, and in this respect they all deviate from man. In the chimpanzee [1] it reaches the posterior aspect of the disc between the first and second lumbar vertebræ, and approaches somewhat the condition which is found in the human infant. It must be borne in mind, however, that the specimen which was examined was probably not more than four years old. In the gibbon the spinal cord is relatively longer than in the chimpanzee, and extends downwards to the level of the middle point of the third lumbar vertebra. In the orang it is remarkable for its shortness. In my specimen it ended in the usual conus medullaris opposite the lower border of the eleventh dorsal vertebra.

But certainly the feature in the spinal cord of the anthropoid apes which is most calculated to arrest attention is the condition of the

[1] The first lumbar vertebra in the chimpanzee corresponds to the second lumbar vertebra in man.

cervical and lumbar enlargements. The former exhibits a development which corresponds exactly with the length of the upper limbs. In the gibbon, therefore, the cervical swelling attains its maximum dimensions; in the orang it is also of great proportionate size; whilst in the chimpanzee it is considerably reduced, but still distinctly larger than in man.

On the other hand, the lumbar swelling in the anthropoid apes is relatively much smaller than in man, as might be expected from the meagre proportions of the lower limbs in the apes.

Mouth, Tongue, and Larynx.—In the mesial section of an anthropoid ape, the elongated form of the mouth, and, as a consequence of this, the quadrupedal shape of the tongue, constitute a very prominent feature. In this respect the gibbon and chimpanzee deviate more from man than the orang: in the last of these the arrangement of the lingual papillae is very similar to what we find in the case of the human tongue.

Quite recently Dr. Symington[1] has called attention to the change of position which the human larynx undergoes in the transition from the infantile to the fully developed adult condition. In the adult the lower border of the larynx corresponds to the intervertebral disc between the sixth and seventh cervical vertebræ; in the newly-born child it is placed as high as the middle or lower border of the fourth cervical vertebra. Between infancy and adult life, therefore, the larynx undergoes a very manifest descent. Symington considers that there is a downward movement of the entire larynx, and he believes that this is caused by the growth of the facial portion of the skull. In adopting this view of the case, he has overlooked the fact, that the upper border of the larynx, as represented by the epiglottis, does not descend to an equal extent with the inferior border as represented by the lower margin of the cricoid cartilage. In the newly-born child (as may be noted in two mesial sections of the entire frozen body in the Trinity College Anatomical Department) the tip of the epiglottis corresponds with the disc of cartilage which separates the odontoid process from the body of the

[1] "Journ. Anat. and Phys.," 1885.

axis vertebra. In the adult its position may be studied in the mesial sections of a male and a female represented by Braune: in the former it is placed opposite the upper border of the third cervical vertebra, and in the latter opposite the lower border of the same vertebra. It follows from this that the upper border of the larynx descends a distance corresponding to the depth of the bodies of two cervical vertebræ with the disc intervening between them; whilst its lower border travels a distance corresponding to the depth of two and a-half vertebral bodies and two intervertebral discs.

But the face in the anthropoid apes is much greater than that of man, and its development in the transition from the infantile to the adult condition is still more striking. If the growth of this part of the skull, therefore, pushes down the human larynx, we would naturally expect the displacement to be more marked in the apes than in man. So far from this being the case, we find in the former a more infantile condition of affairs. In the chimpanzee and orang the tip of the epiglottis is placed opposite the junction between the odontoid and the body of the axis vertebra; whilst the lower border of the cricoid cartilage lies opposite the intervertebral disc between the fifth and sixth cervical vertebræ.

Another point which must enter into our calculations in considering this matter is, that in the newly-born child the cervical region of the vertebral column is proportionally longer than the same segment in the adult, and therefore its subsequent growth takes place at a slower pace than in the case of the dorsal and lumbar regions.

I am inclined to believe that the alteration in the position of the larynx which accompanies the growth of a child is very largely due to the organ growing more in a downward than in an upward direction; and, further, that the entire displacement, which is evident from the sinking downwards of the epiglottis, is in all probability due to the vertebral column outstripping the larynx in its upward growth. It is difficult to conceive how the growth of the face could materially affect the position of the larynx.

The laryngeal pouch in the chimpanzee, as will be seen in Plates VII. and VIII., extends downwards in front of the sternum to the lower border

of the manubrium; it stretches in an upward direction until it abuts against the hollow posterior surface of the hyoid bone. In Plate VII. a glass rod is represented as passing from the larynx into this pouch, and in Plate VIII. the narrow throat of communication is seen in section.

In the orang the laryngeal pouch, although it was prolonged down to the top of the sternum, was not continued in front of it, as will be observed in the outline tracing in Plate III.

Fig. 9.

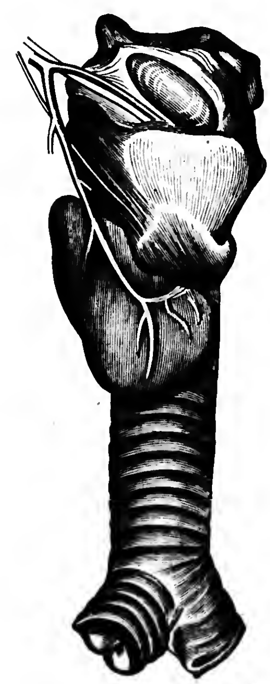


Fig. 10.

The two sides of a human larynx preserved in the Anatomical Department of Trinity College, Dublin, in which the laryngeal ventricles show an unusual degree of development. This specimen was described by Dr. E. H. Bennett in the "Trans. Path. Soc. of Ireland."

I have introduced two woodcuts (Figs. 9 and 10), to show the occasional enlargement which occurs in the laryngeal ventricles of man.

The specimen which these represent was described by Dr. E. H. Bennett in the "Transactions of the Pathological Society of Ireland," and I am indebted to him for the use of the wood-blocks.

Thoracic and Abdominal Viscera.—It is well known to anatomists that the topographical anatomy of the thoracic and abdominal viscera in the infant and child is very different from that of the adult. If we take the vertebral column as our standard, the viscera of the infant are placed at a higher level, and they undergo a gradual subsidence as life advances. Such being the case, it has occurred to me that it might be of interest to compare the relative positions of the viscera in man and the anthropoid apes. By this means we should be able to arrive at a conclusion as to whether the erect attitude which has been assumed by man has led to any material displacement of the viscera in their relation to the vertebral column.

The representations which are given by Braune of mesial sections of a male and a female will serve us for purposes of comparison in so far as the adult man is concerned: the facts relating to the newly-born infant I have obtained from two sections (of a male and a female) which I have made myself.

On comparing the various sections, I was surprised to find that in the position of the thoracic and abdominal viscera there is very little difference between the chimpanzee and the adult man; and this is all the more remarkable, when we bear in mind that the chimpanzee with which the comparison was instituted was far from being a mature specimen.

In the orang we observe a different state of affairs. The viscera hold positions, in reference to the vertebral column, almost identical with those of the viscera in a newly-born child. Thus we find the bifurcation of the trachea taking place in both at the level of the third dorsal vertebra; in both the arch of the aorta rises as high as the second dorsal vertebra, which in the orang carries it above the level of the manubrium sterni; in both the lower surface of the heart is placed at a level corresponding to the body of the seventh dorsal vertebra. But it is needless to enumerate in detail other points of similarity, as these can be more easily appreciated by an examination of the following Table:—

POSITION OF VISCERA IN RELATION TO VERTEBRAL COLUMN IN MAN AND THE ANTHROPOID APES.

	Adult Male (Braune).	Adult Female (Braune).	Chimpanzee.	Newly-born Infant.		Orang.	Gibbon.
				Male.	Female.		
Tip of the Epiglottis, ...	C. III. (upper border).	C. III. (lower border).	Junction of Odontoid with C. II.	Junction of Odontoid with C. II.	Junction of Odontoid with C. II.	Junction of Odontoid with C. II.	Middle point of C. II.
Lower border of Cricoid Cartilage, .	Disc between C. VI. & C. VII.	C. VII. (upper border).	Disc between C. V. & C. VI.	C. IV. (lower border).	C. IV. (lower border).	Disc between C. V. & C. VI.	—
Bifurcation of Trachea, ...	D. IV.	D. IV.	D. V.	D. III.	D. III.	D. III.	—
Summit of Aortic Arch, ...	D. IV.	D. IV.	D. III.	D. II. (upper border).	D. II.	D. II.	—
Right Pulmonary Artery, ...	D. V.	Disc between D. V. & D. VI.	Disc between D. V. & D. VI.	D. IV.	D. III.	D. IV.	—
Lower Surface of Heart, ...	Disc between D. VIII. & D. IX.	Disc between D. IX. & D. X.	Disc between D. IX. & D. X.	Disc between D. VII. & VIII.	D. VII.	D. VII.	D. VIII. & D. IX.
Upper border of Manubrium Sterni, .	D. III.	D. II.	D. II.	D. I.	C. VII.	D. III.	Disc between D. I. & D. II.
Stomach, ...	L. I. & L. II.	L. I.	D. XIII. & L. I.	D. XII. & L. I.	L. I. & L. II.	D. XI. & D. XII.	D. XI., D. XII., & D. XIII.
Duodenum, ...	L. III.	L. II. & L. III.	L. II.	L. II. & L. III.	L. II.	L. I. & L. II.	—
Pancreas, ...	D. XII., L. I., & L. II.	D. XII., L. I., & L. II.	D. XIII., L. I. & L. II.	L. I. & L. II.	L. I. & L. II.	D. XI., D. XII. & L. I.	D. XII. & D. XIII.
Bifurcation of Aorta, ...	L. IV.	L. IV.	L. IV.	—	—	Disc between L. III. & L. IV.	L. IV.

C = cervical vertebra. D = dorsal vertebra. L = lumbar vertebra.

Pelvic Viscera.—In Plate IX. two views are given of the human pelvis in mesial section. Fig. 1 is the pelvis of a newly-born child, and Fig. 2 that of an adult. They are introduced in order that they may be compared with the corresponding region in the chimpanzee, as exhibited in Plates VII. and VIII. A striking difference exists in the position of the pelvic viscera in the adult man and the anthropoid apes. The rectum in the chimpanzee is almost straight; in man it pursues a curved course, and its terminal portion is bent backwards. In these respects the newly-born child stands midway between the anthropoid apes and the adult man; the rectum is much straighter than it is later in life, and the terminal portion proceeds almost vertically downwards.

These differences are largely due to the degree of obliquity of the pelvis, and to the degree of curvature of the sacrum and coccyx. As is well known, a long distance intervenes in the anthropoid between the tip of the coccyx and the anus, so that the rectum and bladder are to a great extent unprotected on the posterior aspect of the body. It is interesting to note that the distance between the tip of the coccyx and the anus in the new-born child (as made out by measurements on the frozen body) appears to be relatively greater than in the adult man.

APPENDIX.

In the first section of Part I. of this Memoir the form adaptation of the bodies of the lumbar vertebræ, with reference to the lumbar curve, is alone considered, and it has been noted that the bones contribute very little to the curve. In the following Table the part which the intervertebral discs play in the construction of the curvature is given in five spines (three females and two males), and indices have been calculated for these upon the same method as we have previously adopted for the vertebral bodies (p. 4). It will be noticed, as indeed might have been foreseen, that a low lumbo-intervertebral index is always accompanied by a high lumbo-vertebral index, and *vice versâ*. As a general rule the indices of the vertebræ diminish in the European in a regular manner from the first to the fifth lumbar vertebra; but this is not the case with the indices of the discs. A considerable amount of irregularity is observed in these.

INDICES OF THE VERTEBRAL BODIES AND INTERVERTEBRAL DISCS OF THE LUMBAR REGION IN FIVE IRISH SPINES.

	Female Spine.			Female Spine.			Female Spine, No. 3, Plate 2.			Male Spine, No. 1, Plate 2.			Male Spine.		
	Actual depth in Millimetres.		Index (anterior depth = 100).	Actual depth in Millimetres.		Index (anterior depth = 100).	Actual depth in Millimetres.		Index (anterior depth = 100).	Actual depth in Millimetres.		Index (anterior depth = 100).	Actual depth in Millimetres.		Index (anterior depth = 100).
	Front.	Back.		Front.	Back.		Front.	Back.		Front.	Back.		Front.	Back.	
I. Lumbar Vertebra, . .	26·0	27·5	105·7	25·5	27·0	105·8	24·0	23·5	97·9	26·5	30·0	113·2	25·0	26·5	106·0
I. Lumbar Disc, . . .	8·0	4·0	50·0	8·0	5·0	62·5	8·0	7·0	87·5	7·0	7·0	100·0	7·0	3·5	50·0
II. Lumbar Vertebra, . .	28·5	28·0	98·2	27·0	27·0	100·0	24·5	24·5	100·0	29·0	29·0	100·0	27·0	28·0	103·7
II. Lumbar Disc, . . .	7·5	4·5	60·0	9·5	7·0	73·7	10·0	9·0	90·0	8·0	9·0	112·0	8·0	6·0	75·0
III. Lumbar Vertebra, . .	29·0	27·5	94·8	27·0	27·0	100·0	25·0	23·0	92·0	29·5	26·5	89·8	26·0	25·0	96·1
III. Lumbar Disc, . . .	8·0	5·5	69·0	8·0	4·0	50·0	11·0	8·0	72·7	9·5	8·5	89·4	10·0	6.5	65·0
IV. Lumbar Vertebra, . .	29·5	27·0	91·5	27·0	26·0	96·3	26·0	23·0	88·4	28·5	25·0	87·7	25·0	25·5	102·0
IV. Lumbar Disc, . . .	8·5	6·0	70·6	8·5	7·5	88·2	10·0	9·0	90·0	9·5	9·5	100·0	9·0	5·0	55·5
V. Lumbar Vertebra, . .	30·0	25·0	80·3	26·0	24·0	92·3	26 0	20·0	76·9	29·0	23·0	79·3	26·5	22·0	83·0
V. Lumbar Disc, . . .	12·0	3·5	29·1	17·5	6·0	34·2	13·0	8·0	61·5	16·5	8·0	48·4	16·5	6·0	36·3
Lumbo-vertebral Index, . .	..	..	94·4	..	..	98·4	..	..	90·8	..	..	93·6	..	..	98·1
Lumbo-intervertebral Index, .	..	..	54·1	..	..	57·4	..	..	79·0	..	..	83·1	..	..	53·8

EXPLANATION OF THE PLATES.

PLATE I.

Charts to show, in a graphic manner, the form-adaptation of the lumbar vertebræ to the lumbar curve.

Chart A exhibits the form-adaptation of the lower five true vertebræ in the chimpanzee, gorilla, orang, Australian, Andaman, and European. The dotted lines refer to the apes, and the solid lines to man. For fuller explanation see p. 7.

Chart B shows the sex-differences in the form-adaptation of the lumbar vertebræ to the lumbar curve in the Irish, Tasmanian, Australian, Andaman, and Negro. For further explanation see p. 30. In this chart the dotted lines refer to the females, and the solid lines to the males.

Chart C brings out the sex-differences in the form-adaptation of the lumbar vertebræ to the lumbar curve in the Irish, Australian, and Andaman, see. p. 30. The dotted lines indicate the females, and the solid lines the males.

PLATE II.

This plate represents a series of reduced tracings of mesial sections of the human spine. Two of these, viz. No. 2 (male) and No. 5 (female), have been taken from Professor Braune's *Atlas of Topographical Anatomy*. Another adult male spine is a reproduction of the tracing which is given by the Brothers Weber in their "Mechanik der menschlichen Gehwerkzeuge" (1836). This spine was imbedded in plaster of Paris, and then divided in the mesial plane. The outlines of the spine of the girl, *æt.* 13, and that of the boy, *æt.* 6, were obtained from two drawings in an important work upon the *Topographical Anatomy of the Child* which will shortly be published by Dr. Symington of Edinburgh.

The remaining five representations of male spines, and four representations of female spines, are reduced tracings of mesial sections of the frozen and isolated vertebral column. In every case the tracing was taken when the spine was still in the frozen condition, and the reduction effected by photography, so as to obtain absolutely reliable results.

By means of the line which intersects the lumbar region the degree of lumbar prominence may be estimated. The indices of curve, which are obtained by comparing the distance of the most projecting point in front of this line with the length of the lumbar segment of the column, as well as the indices of the vertebræ of these spines, &c., &c., are given in Table R, p. 26.

PLATE III.

This plate shows the spinal curvature in two newly-born children, four chimpanzees, and one orang-utan.

Full-time human fœtus (*male*).—A mature child, still-born. It was placed on its back, and the lower limbs were allowed to assume the position most natural to them: in other words, the thighs remained semiflexed, and everted. The chin was raised slightly from the chest. In this position the child was frozen, and then divided in the mesial plane. The tracing was taken when the preparation was still in the frozen state, and it was reduced by photography. The raising of the head has produced a feeble cervical curve; the dorsal and lumbar regions form a continuous concavity, somewhat flattened, from the child having been placed on its back. The promontory is very feeble, and the curve of the sacrum by no means conspicuous.

Full-time human fœtus (*female*).—This is a reduced tracing of a full-time still-born child. It was frozen on its back; but in this case the head was allowed to remain flexed upon the chest, while the lower limbs were straightened. The thighs were placed close together, and then the knees were pressed forcibly backwards until they came in contact with the surface upon which the child was lying. In this position they were fixed. The tracing was taken from the frozen child after it had been divided in the mesial plane, and it was reduced by photography. In this case there was no cervical curve, because the head was flexed; but the straightening of the limbs has produced a strong lumbar convexity—a curve, indeed, which is stronger than in an adult, in the proportion of 11 to 9. Further, the pelvis has been tilted back until the brim is nearly in a line with the spine. This can be seen from the position of the symphysis pubis.

Troglodytes niger (*male, and female* No. 1) are reduced tracings of mesial sections of a young male and a young female chimpanzee. In both cases the animal was frozen on its back, and the lower limbs placed under no restraint. Unfortunately, the female specimen got twisted in the freezing box, and thus it was impossible to cut the entire length of the spine in the mesial plane.

Troglodytes niger (*female*, No. 2) represents a mesial section of headless and flayed carcass of a very young chimpanzee which I obtained from Dr. Wm. Frazer, of Dublin.

Troglodytes niger (*female*, No. 3) represents the outline of the anterior face of the spine of a female chimpanzee, obtained by moulding a strip of ductile lead upon the front of the vertebral column. The dotted line intersects the outline at the level of the lower border of the fourth lumbar, and of the upper border of the thirteenth dorsal vertebra.

All these tracings of the chimpanzee were reduced by photography.

Simia satyrus (*female*).—This is the tracing of a mesial section of an orang-utan, reduced by photography. It was frozen on its back, and no restraint was put upon the lower limbs. The outline of the brain, and of some of the other organs, is given. The relative position of cerebrum to cerebellum is seen; the shortness and the forward position of the corpus callosum

is manifest. In the abdomen, the stomach, which is cut close to the pylorus, lies close under the liver; and in the pelvis, the relative positions of bladder, uterus, and rectum may be noted.

PLATE IV.

Tracings of mesial sections of a number of the lower apes reduced by photography. The tracings were taken when the preparations were still in the frozen state.

Macacus nemestrinus, No. 1, was frozen on its back, with the thighs flexed on the abdomen; owing to the head being slightly bent to the side the cervical region was not cut in the mesial plane.

Macacus nemestrinus, No. 2, was frozen on its back; but in this case the thighs were extended. This was the first section which was made, and the outline of the body was omitted to be taken.

Macacus rhesus: frozen on its back, with the thighs flexed on the abdomen.

Cercopithecus campbelli, No. 1, was frozen on its back, with the thighs flexed on the abdomen. No. 2 is the same animal thawed, placed on its side, and the lower limbs extended to the extent indicated in the tracing.

Semnopithecus entellus was frozen on its back, with its thighs flexed on the abdomen. The straightness of the spine in this case is very remarkable.

Colobus vellerosus (white-thighed colobus), No. 1, was frozen on its back, with its thighs flexed upon the abdomen. No. 2 is the same animal thawed, placed on its side, and the lower limb extended.

PLATE V.

Reduced tracings of mesial sections of a number of the lower apes. These were taken when the animals were still in the frozen condition, and the reduction was effected by photography.

Cercopithecus ruber, No. 1, was frozen in the sitting posture in a tall glass jar, in the manner described in p. 98. No. 2 is the same animal after it had thawed, and with the lower limb extended.

Cercopithecus mona, No. 1, was frozen on its back, with the thigh flexed on the abdomen. No. 2 is the same animal after it had thawed, and with the lower limb extended.

Cercocebus fuliginosus, No. 1, was frozen in a plaster-jacket in the sitting posture, in the manner described in p. 99. No. 2 is another specimen of the same species, which was frozen on its back, with the thighs flexed on the abdomen.

PLATE VI.

Tracings of mesial sections of certain of the lower apes reduced by photography.

Cercocebus fuliginosus, No. 3, is the same animal that is figured as No. 2 in Plate V. This tracing was taken after the preparation had thawed, and with the lower limb extended.

Chlorocebus sabaeus, No. 1, was frozen on its back, with the thighs flexed on the abdomen No. 2 is the same animal thawed, and with the limb extended.

Cynocephalus anubis.—This was a large adult baboon, which was frozen on its back with its thighs flexed.

Cebus capucinus was treated in the same manner as the baboon. The curvature of the spine is more quadrupedal in its character than that of any of the other apes examined. Compare it with the tracings of the dog in the following plate (VIa.)

PLATE VIa.

This plate exhibits the curvature of the spine in the quadruped.

Dog.—No. 1. is the tracing of a fox-terrier which was frozen on its back with its limbs unrestrained.

Dog.—No. 2 is a tracing taken from the mesial section of a collie. It was frozen in an attitude approaching the standing posture (p. 108).

Dog.—No. 3 is a tracing of a poodle which was frozen on its back with its hind limbs extended. As it was disturbed in the freezing-box, the anterior portion of the spine was not cut in the mesial plane, and is therefore not represented.

Bear.—The outline of the anterior surface of the spine of a large bear, taken by moulding a strip of ductile lead on the front of the column.

PLATES VII. AND VIII.

These plates are large coloured drawings of the two sides of the mesial section of the male chimpanzee. They were drawn within the tracings, which were taken from the preparation while it was still in the frozen state, and therefore may be regarded as giving accurate representations of the topographical anatomy of this anthropomorphic ape.

PLATE IX.

Fig. 1.—Mesial section of the pelvis of a mature human male fœtus. The entire body was frozen, so that the position of the viscera is natural.

Fig. 2.—Mesial section of the pelvis of an adult human male. The entire body was frozen, so that the viscera might be obtained in their natural positions.

Figs. 1 and 2 are introduced for the purpose of comparison with the pelvis of the chimpanzee, as exhibited in plates VII. and VIII.

Fig. 3 is a drawing from a photograph of the male chimpanzee represented in plates VII. and VIII. The photograph was taken immediately after the section was made by applying the two sides of the animal to each other: it shows the attitude of the limbs and the line of the section. Although the cut is slightly to the one side of the mesial plane in front, it is accurately in the middle line behind.

PLATE X.

Sagittal sections through the heads of a chimpanzee, Cebus capucinus, and man. The section in each case is made in a line with the middle point of the eyeball.

Fig. 1.—Chimpanzee. This drawing was made from a photograph of the section, slightly smaller than the specimen. Anteriorly the section has deviated outwards, so that it does not pass accurately through the middle of the eyeball.

Fig. 2.—Cebus capucinus. Full-size drawing built up within a tracing taken from the frozen surface of the section.

Fig. 3.—Man. This was drawn directly from the specimen, but within a tracing which had first been reduced.

PLATE XI.

Sections through the frozen head of the chimpanzee and orang.

Fig. 1.—Section through the frozen head of the orang in the sagittal direction, and in a line with the central point of the eyeball. The drawing is a full-size representation, and was constructed within the tracing which was taken from the frozen surface.

a. lenticular nucleus.
b. choroid plexus peeping out of the descending horn of the lateral ventricle, which is opened along its outer margin.
c. lateral venous sinus.
d. lateral sinus cut at another point.
e. cochlea.
f. superior maxillary sinus.

Fig. 2.—Sagittal section through the frozen head of a chimpanzee immediately to the outer side of the mesial plane. The tracing was taken from the frozen surface, and the drawing is the size of nature. Only the posterior part of the head is represented.

a. lateral venous sinus.
b. calcarine fissure.
c. corpus dentatum of the cerebellum.
d. margin of the foramen magnum.
e. posterior arch of the atlas.

Fig. 3.—Coronal section of the frozen head of the chimpanzee opposite the widest part of the cerebellum. The tracing was taken when the specimen was still frozen, and the drawing is the size of nature.

a. posterior horn of the lateral ventricle. The dotted line ends upon the bulb of the cornu.

b. hippocampus minor.

c. calcarine fissure.

d. corpus dentatum of the cerebellum.

e. and *f.* lateral venous sinus.

This plate was drawn for me by Dr. St. John Brooks, Demonstrator of Anatomy, Trinity College, Dublin.

PLATE XII.

This plate, which is also the work of Dr. Brooks, represents sections through the frozen gibbon (*Hylobates agilis*—the wauwau).

Fig. 1 is a reduced outline tracing of a mesial section of the gibbon.

Fig. 2 is a full-size drawing of a section through the frozen head of the gibbon. The tracing was obtained when the animal was still in the frozen state.

a. lateral venous sinus.

b. internal parieto-occipital fissure.

g. superior longitudinal sinus.

h. portion of the falx cerebri.

i. anterior pillar of the fornix.

k. anterior commissure.

m. sphenoidal sinus.

n. corpora quadrigemina.

Fig. 3.—Sagittal section through the frozen head of the gibbon, in a line with the central point of the eyeball. The tracing was taken when the specimen was still in the frozen state, and the drawing is the size of nature.

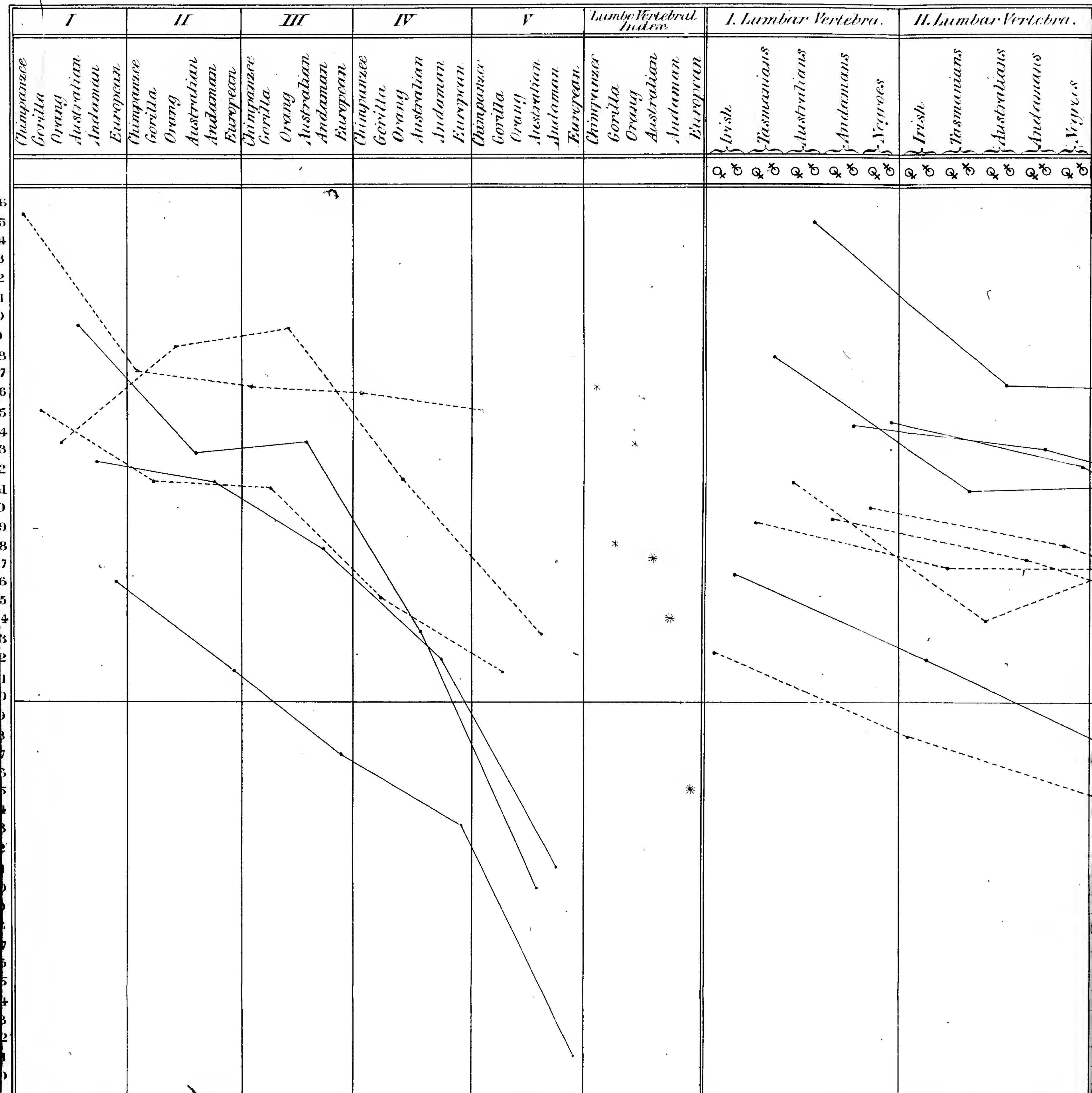
a. lateral sinus.

b. descending cornu of lateral ventricle.

c. cochlea.

d. dentate fissure.

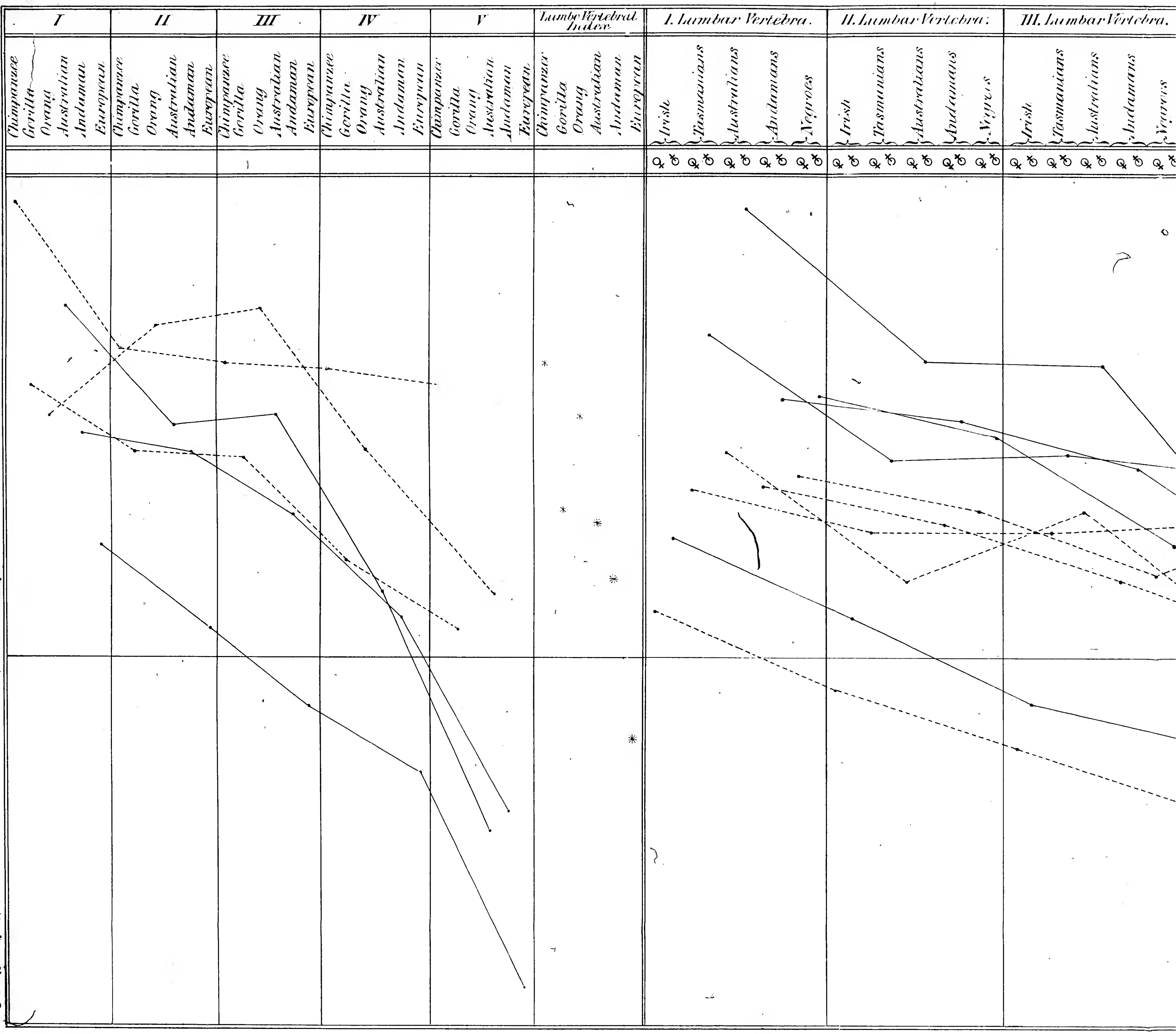
e. lenticular nucleus.


I
II
III
IV
V
Lumbo Vertebral Index
I. Lumbar Vertebra.
II. Lumbar Vertebra.
Chimpanzee
Gorilla
Orang
Australian
Andaman
European
Irish
Tasmanians
Australians
Andamans
Negroes
♀ ♂

D. J. C., del.

A

B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II. Lumbar Vertebra.
III. Lumbar Vertebra.
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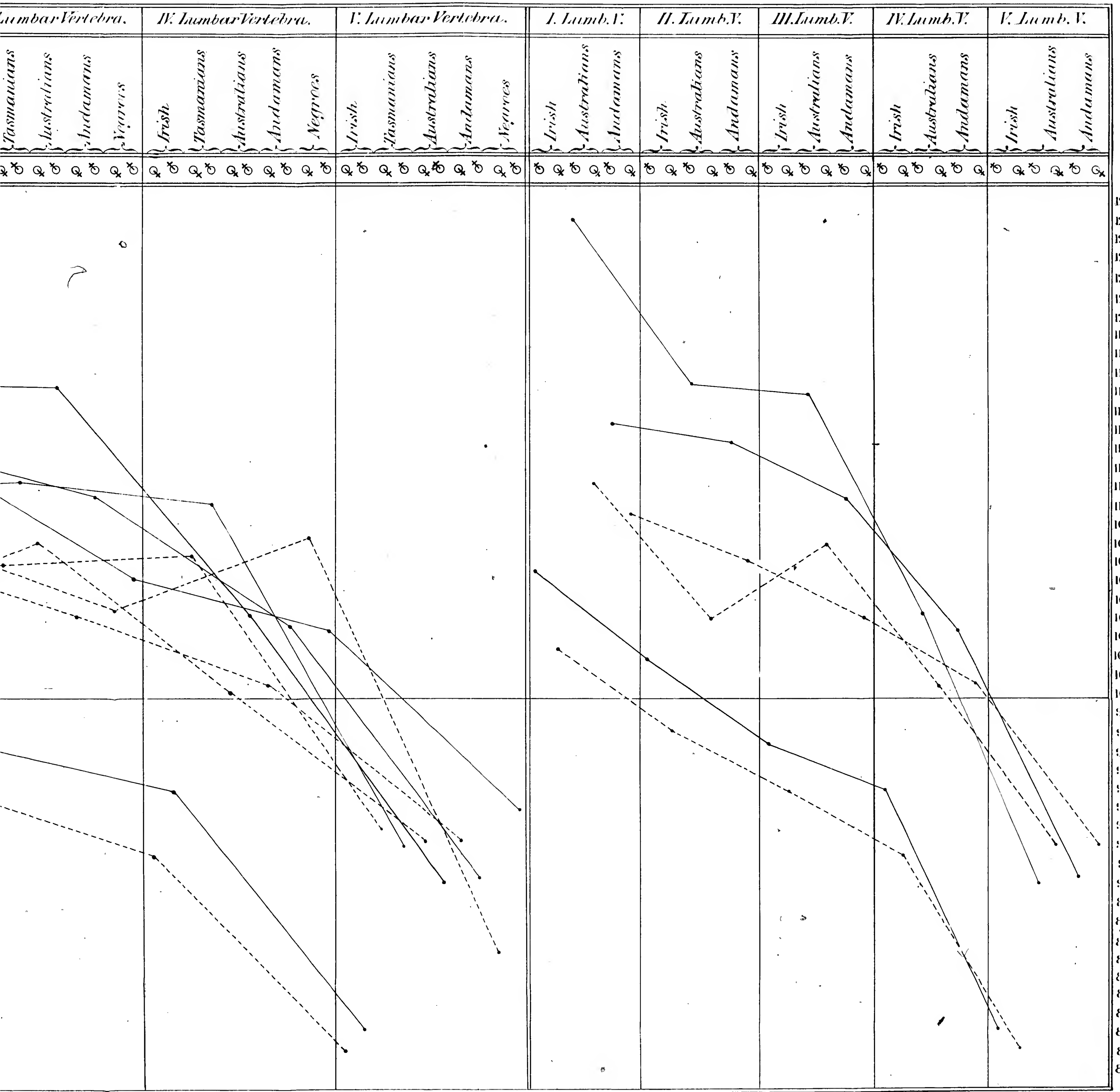
Orang
Australian
Andaman
European
Irish
Tasmanians
Australians
Andamans
Negroes
♀ ♂

D. J. C., del.



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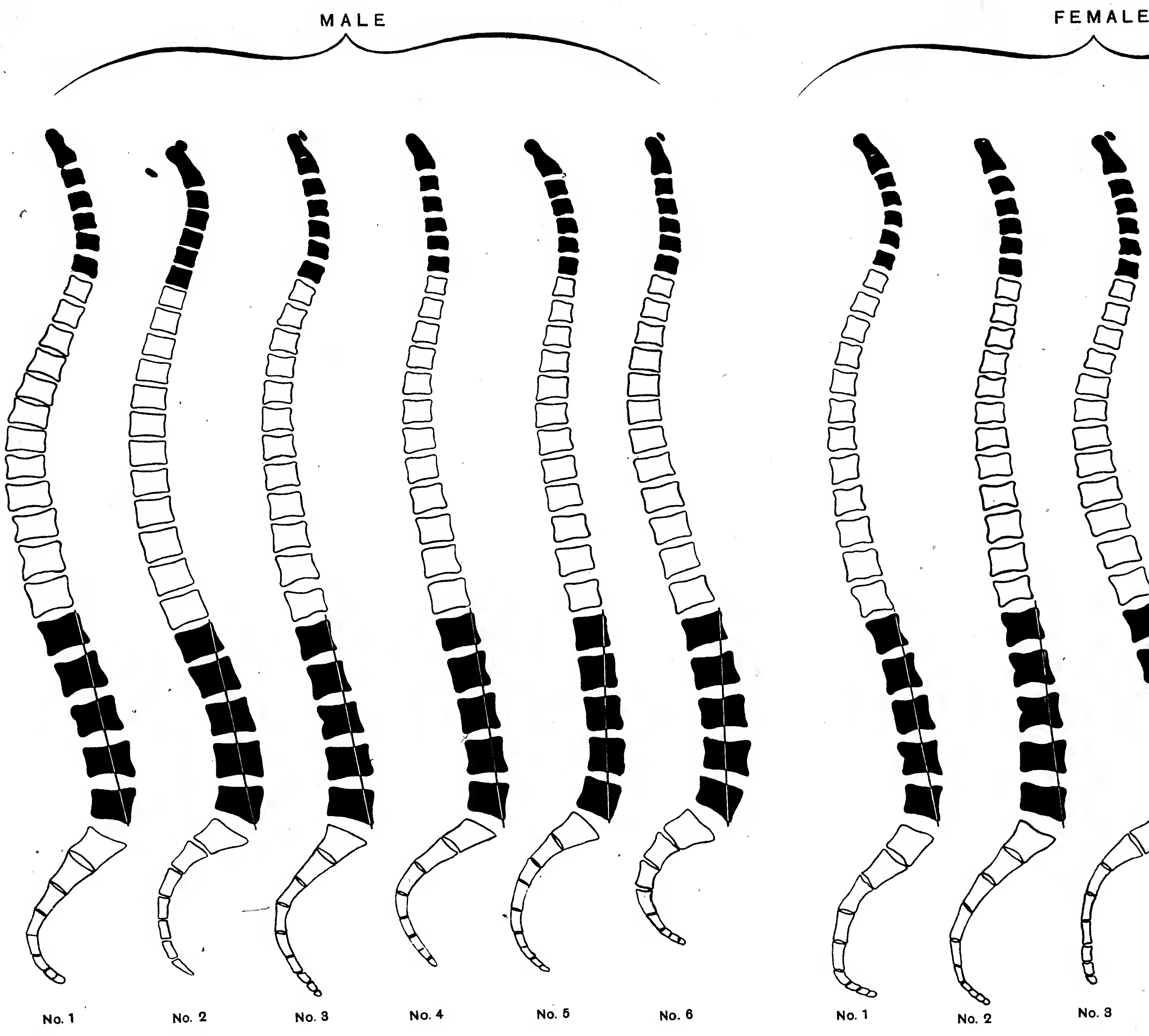

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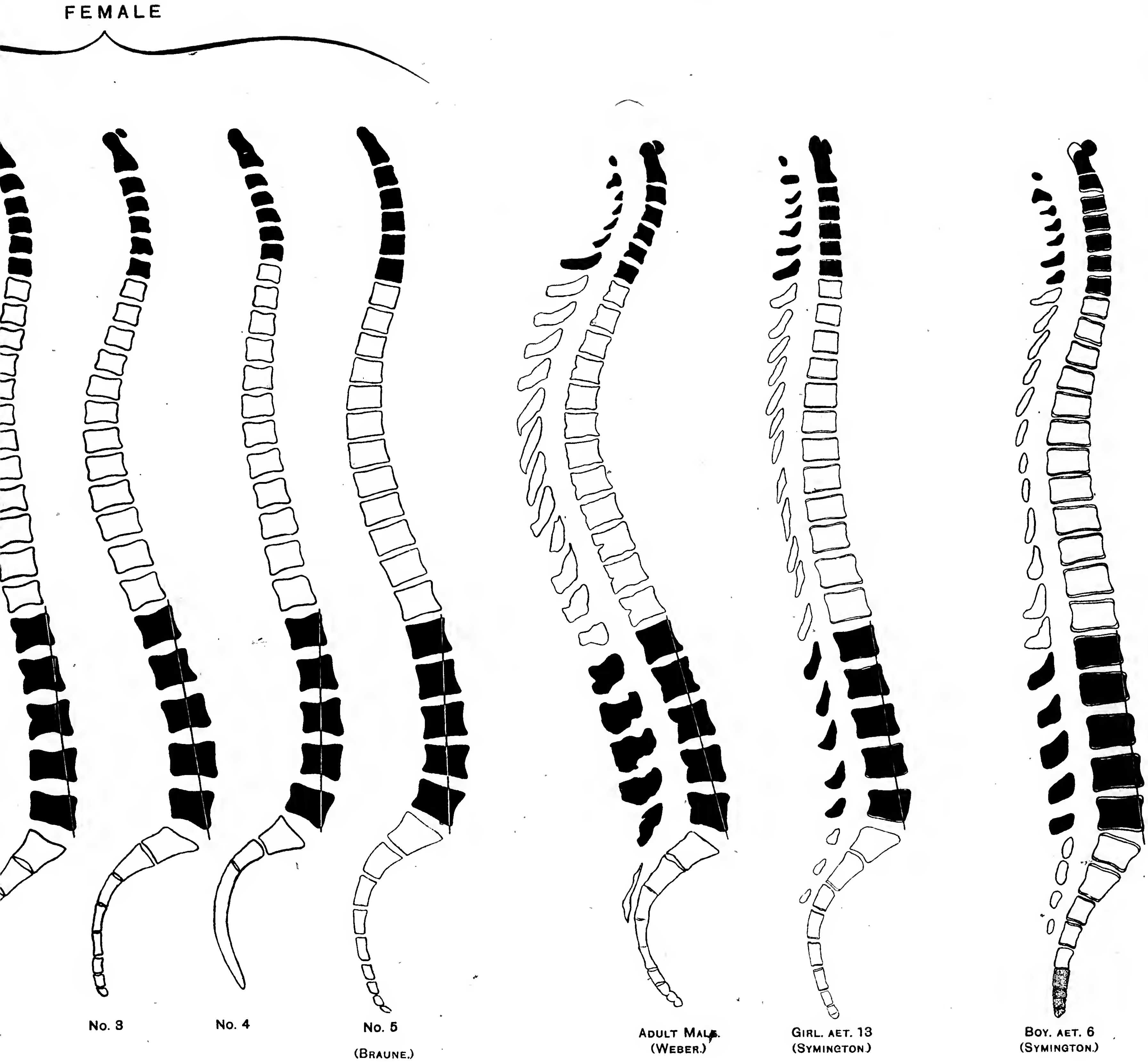
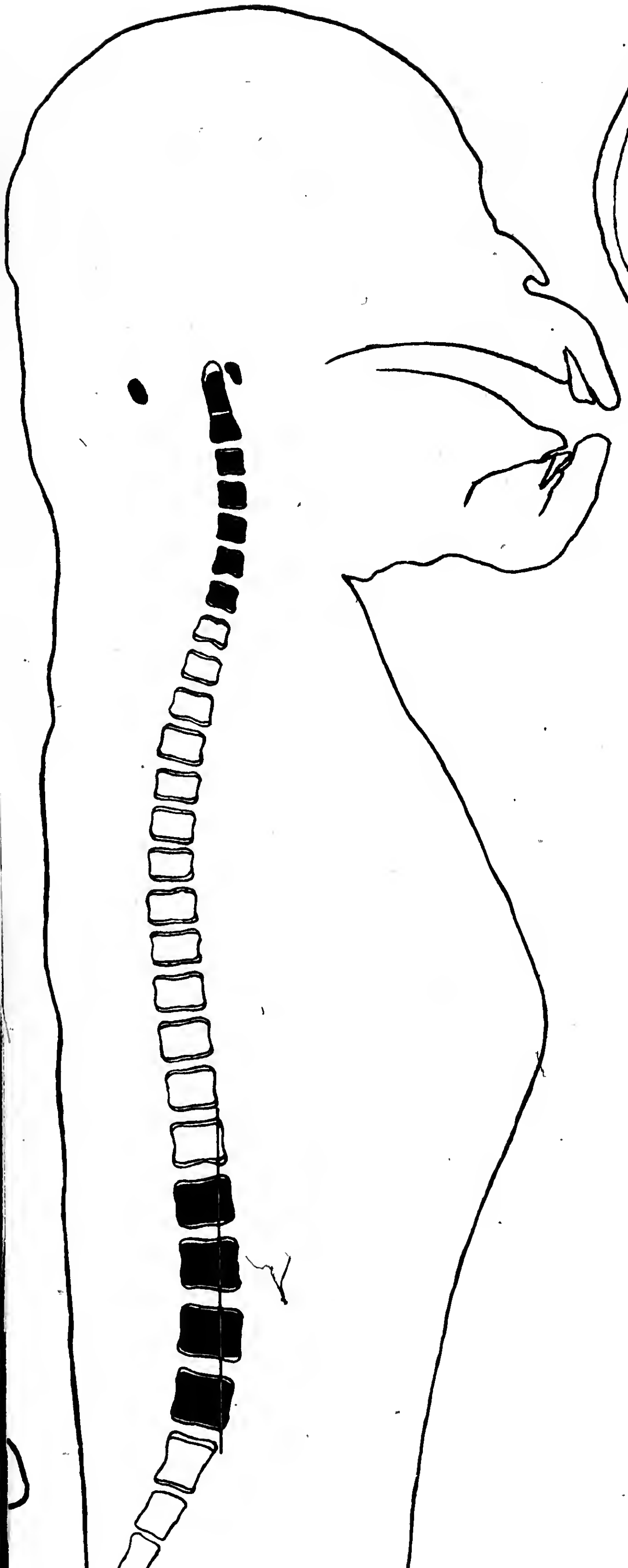
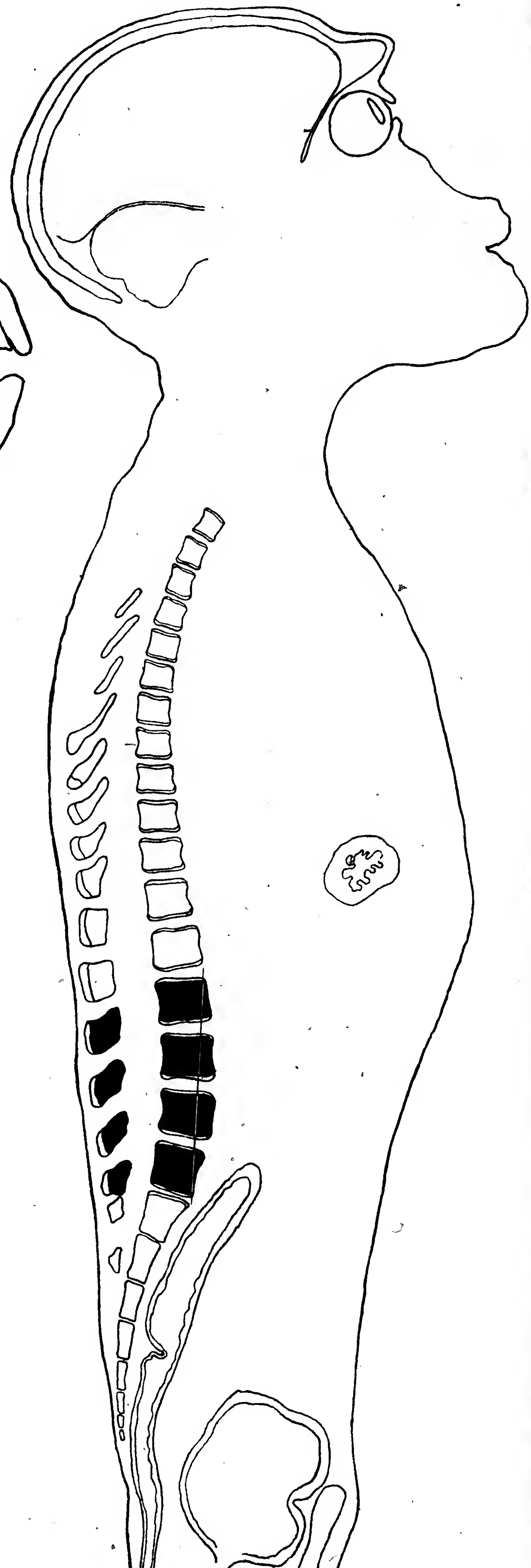


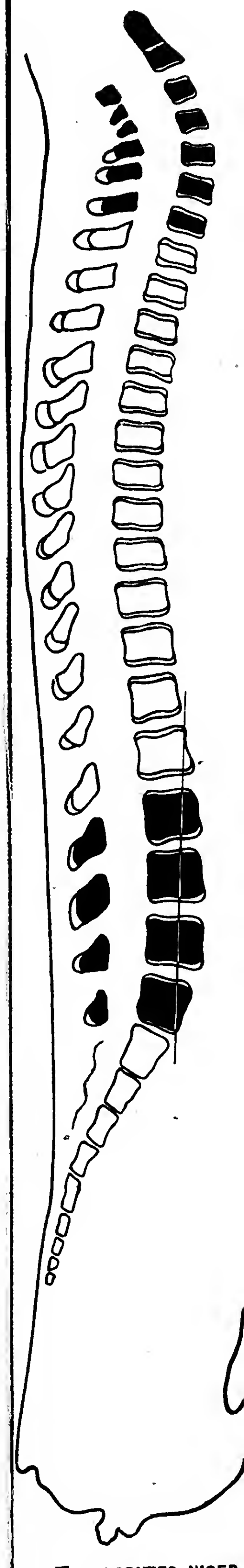
FEMALE
No. 3
No. 4
No. 5
(BRAUNE.)
ADULT MALE.
(WEBER.)
GIRL. AET. 13
(SYMINGTON.)
BOY. AET. 6
(SYMINGTON.)



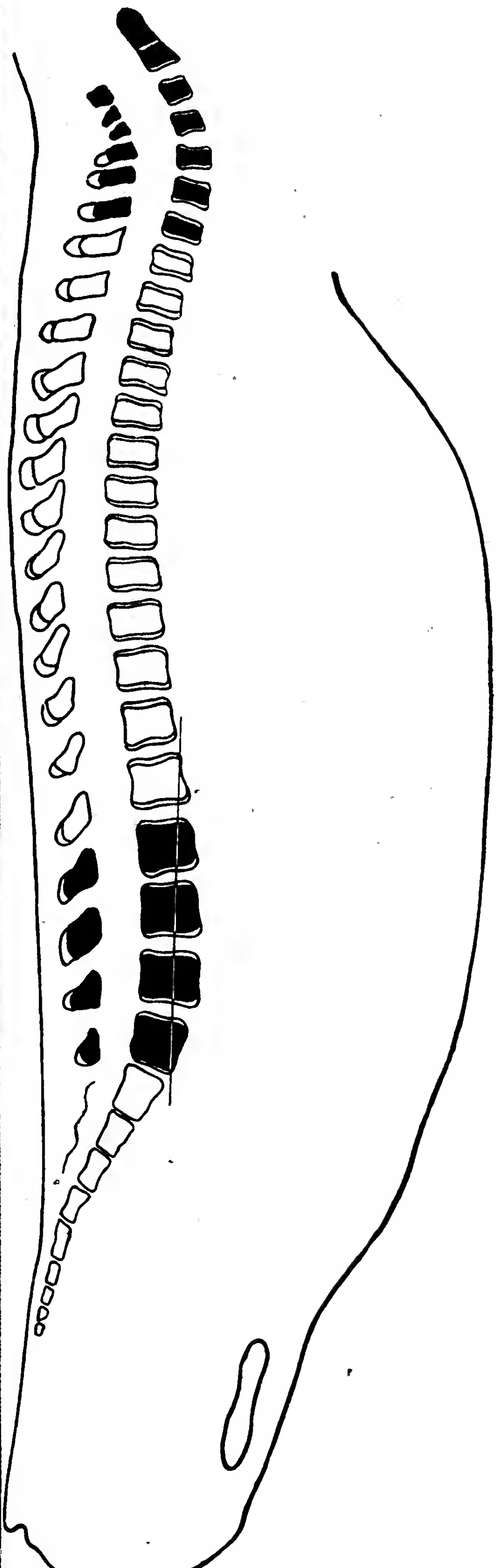
Troglodytes niger
Male



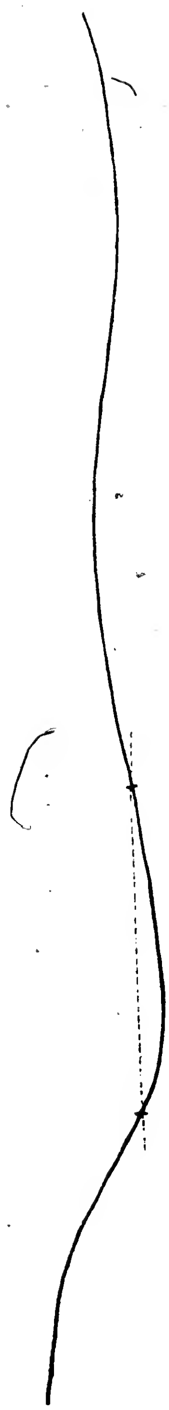
Troglodytes niger
Female No. 1



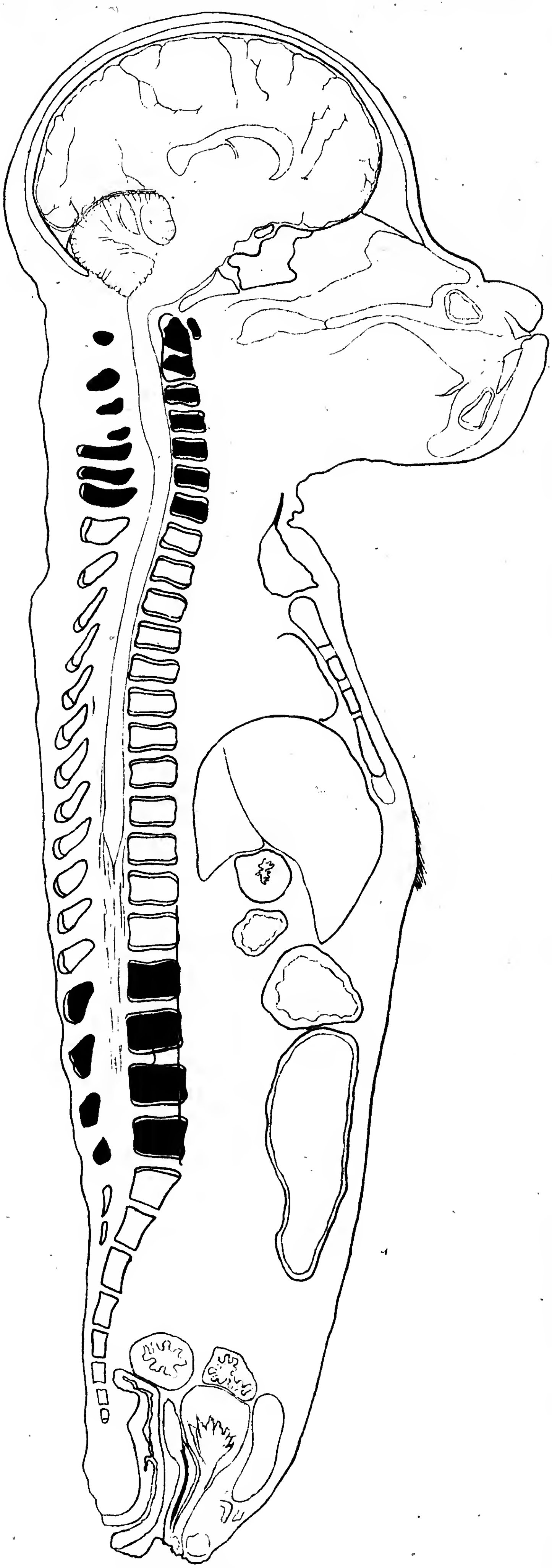
Troglodytes niger
Female No. 2



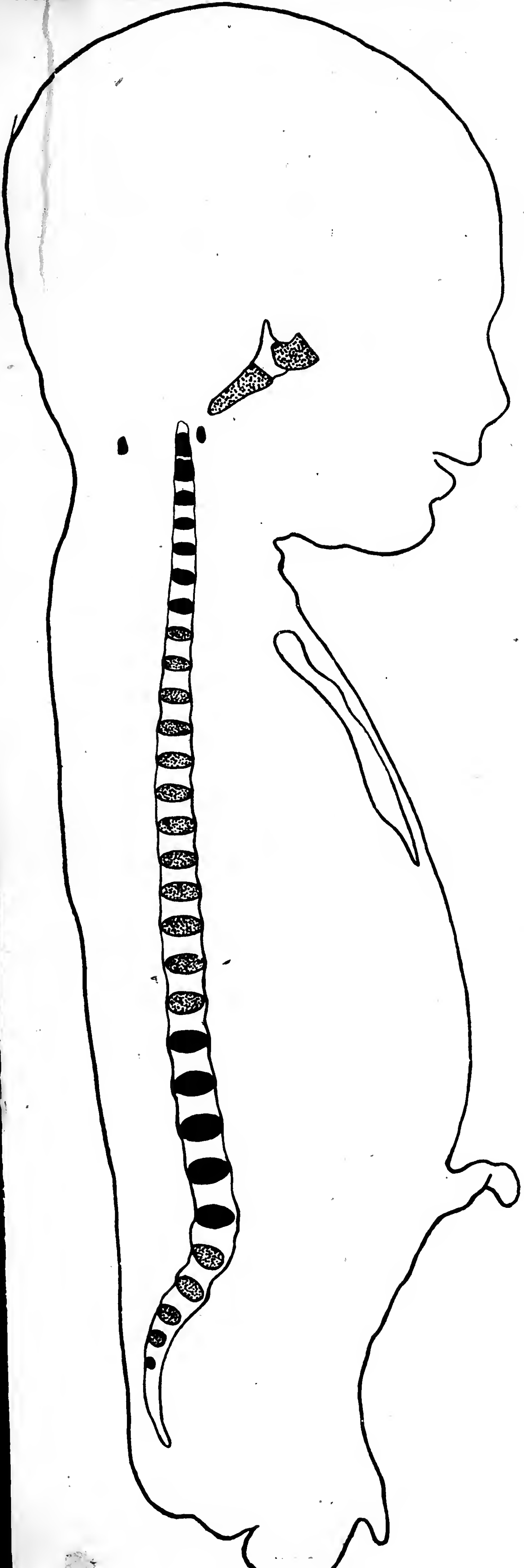
Troglodytes niger
Female No. 2



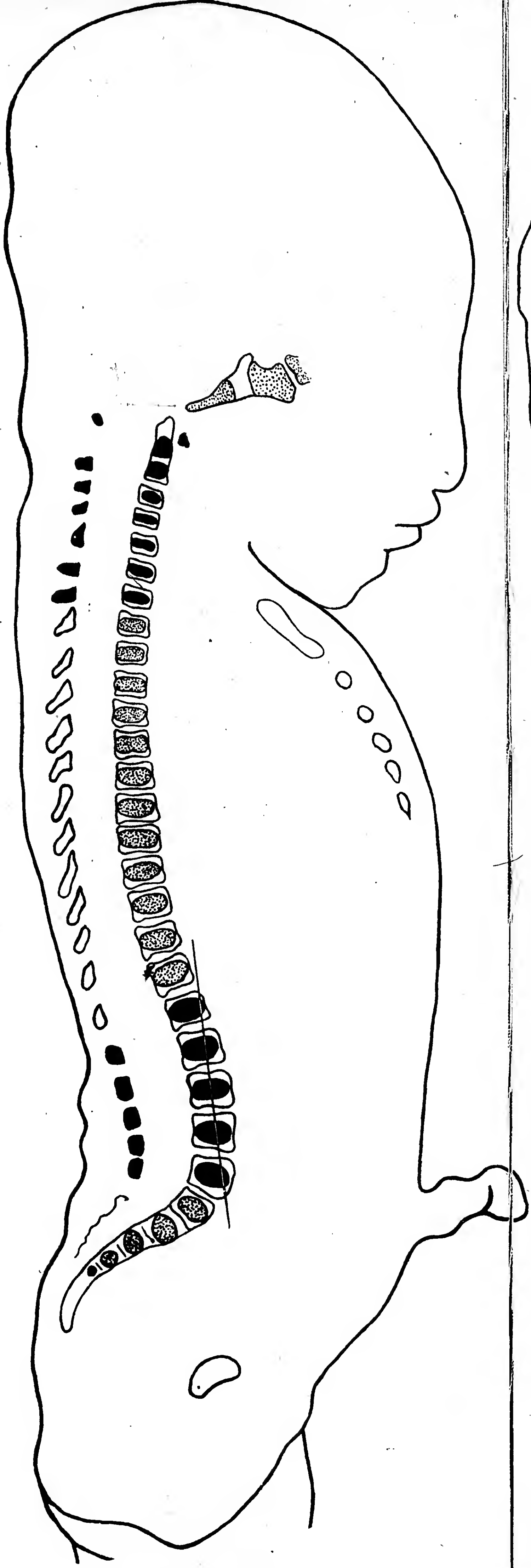
Troglodytes niger
Female No. 3



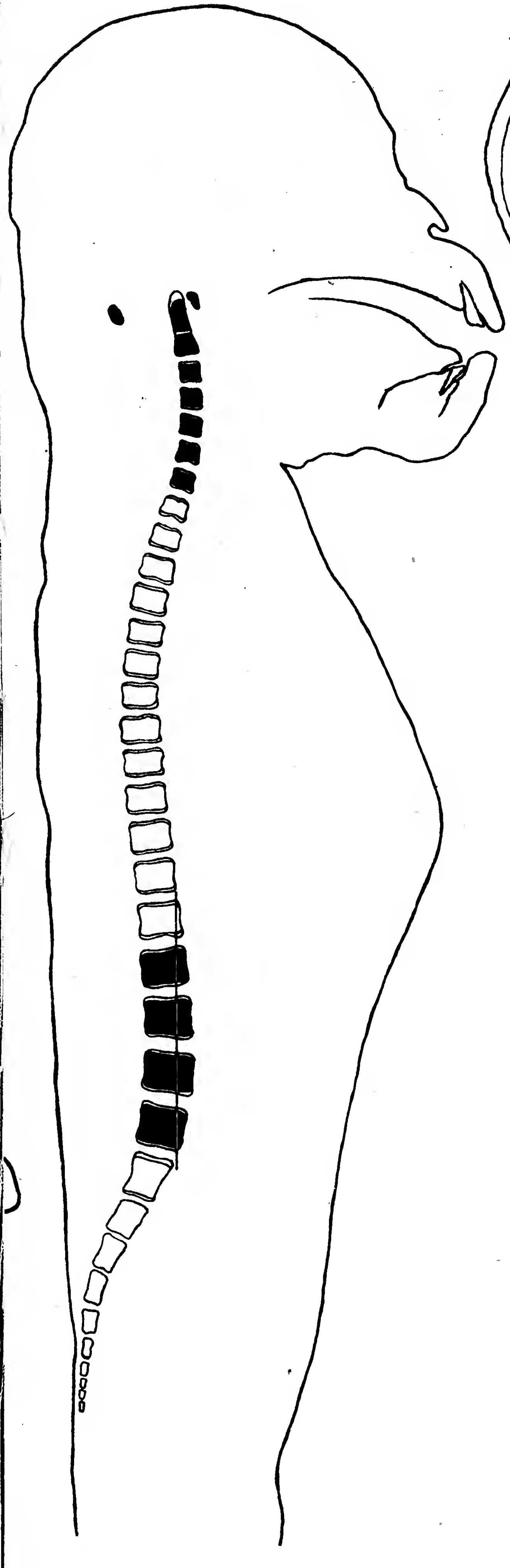
Simia Satyrus
Female



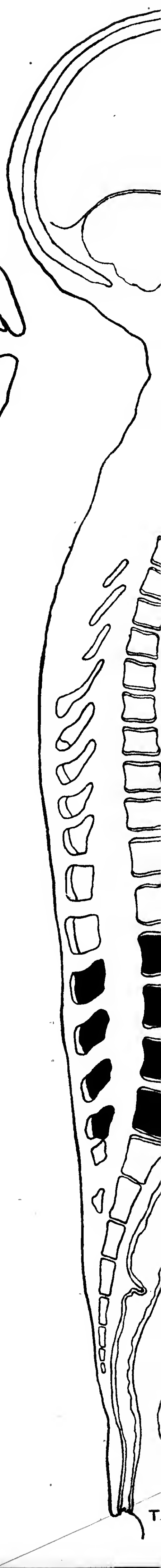
Full Time Human Fœtus.
Male



Full Time Human Fœtus.
Female



Troglodytes niger
Male



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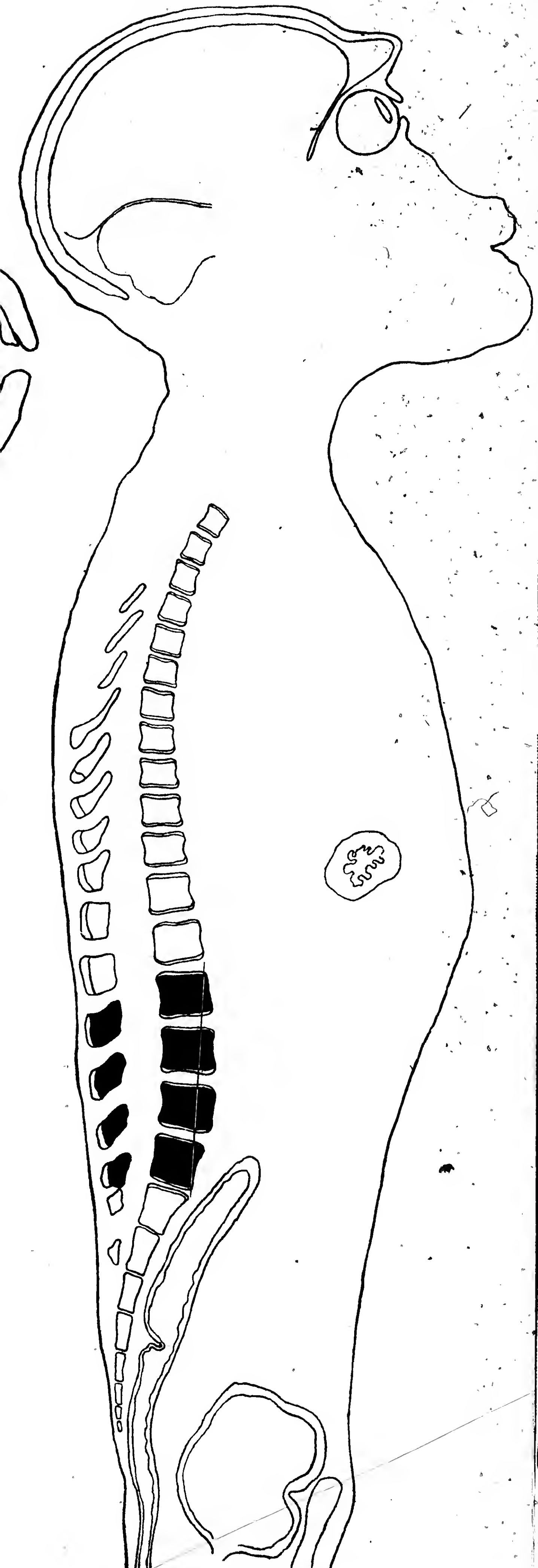
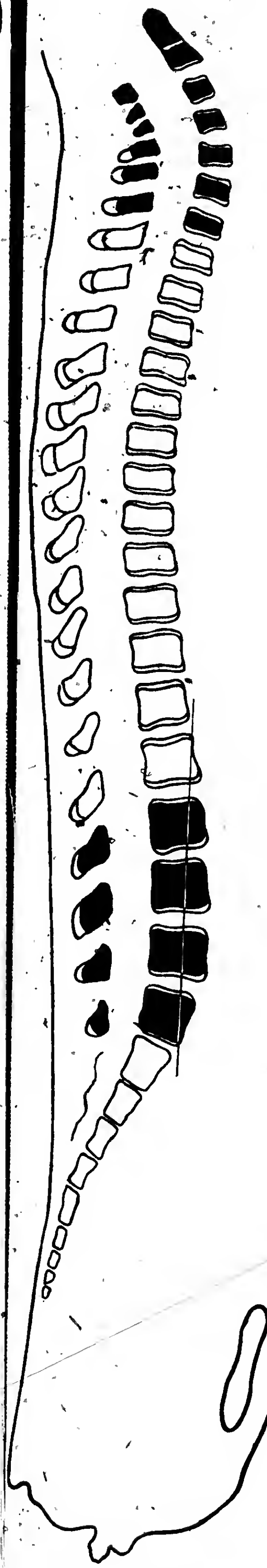
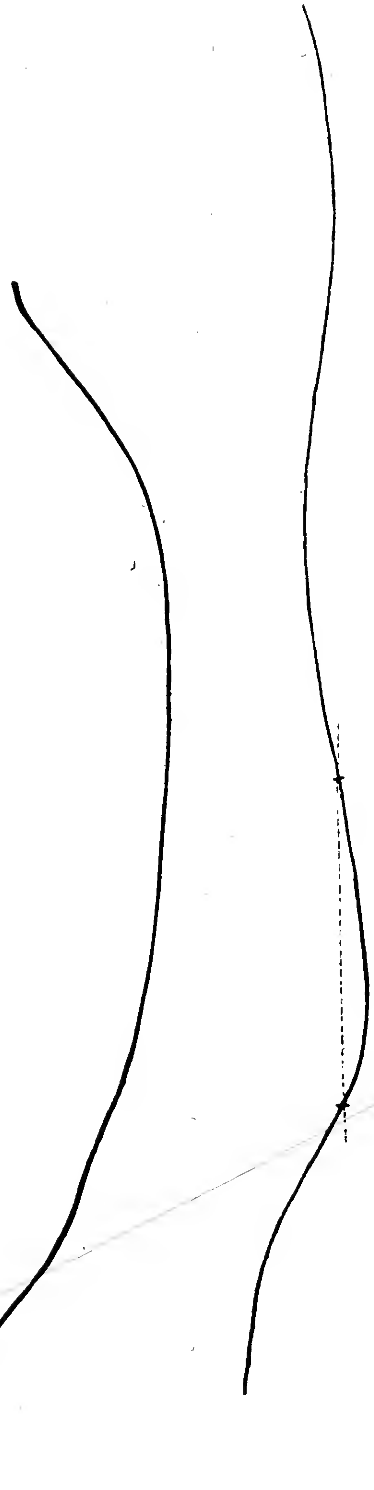


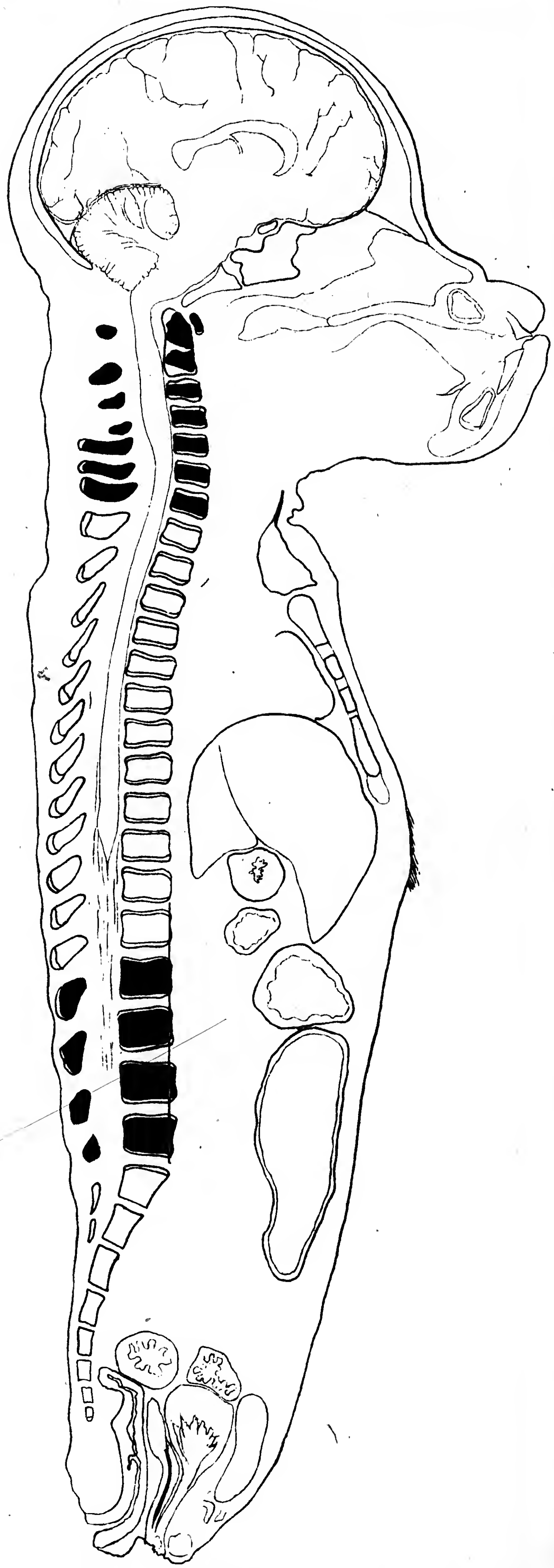
TROGLODYTES NIGER
FEMALE NO. 1



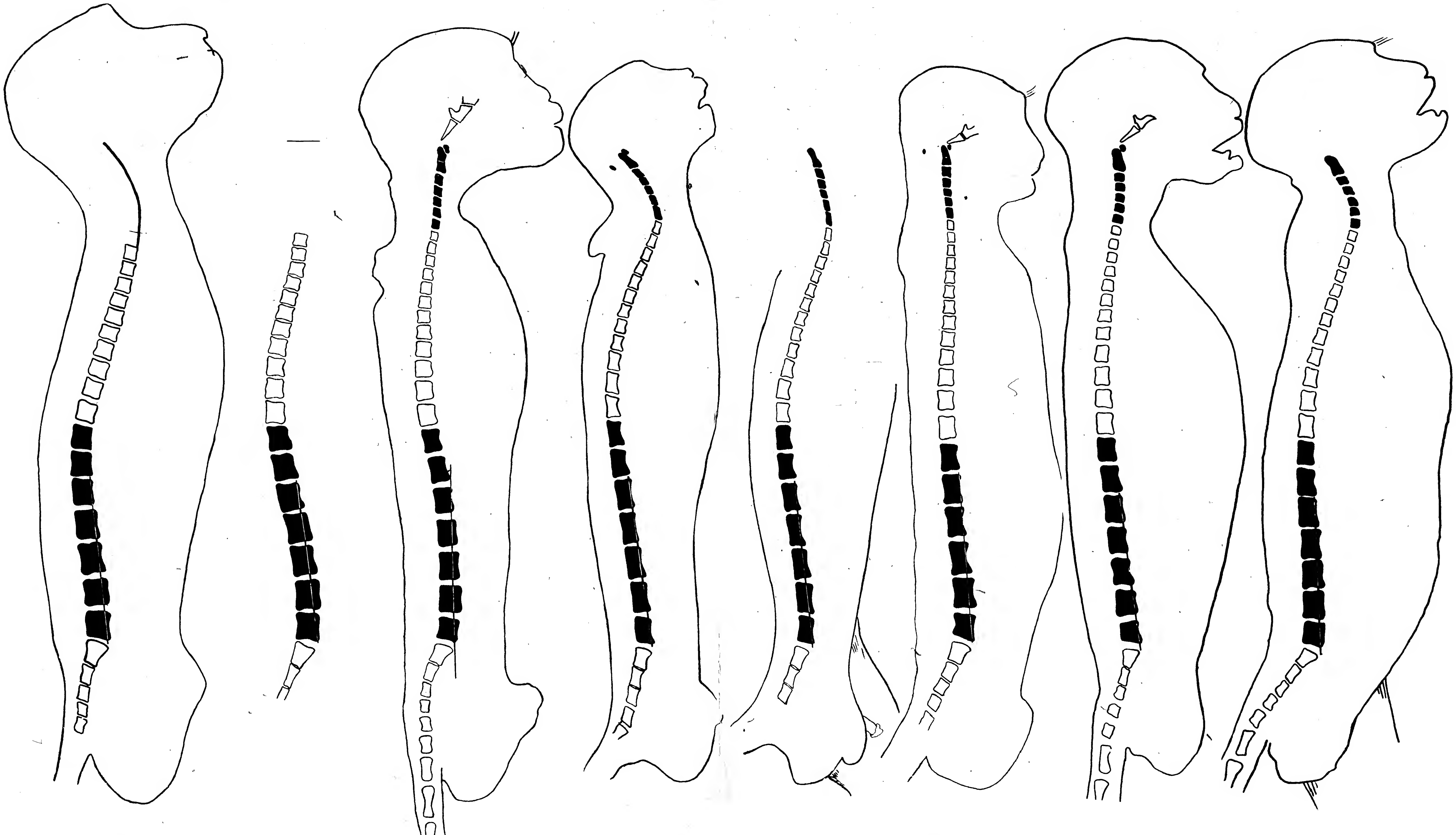
TROGLODYTES NIGER
FEMALE NO. 2



TROGLODYTES NIGER
FEMALE NO. 3

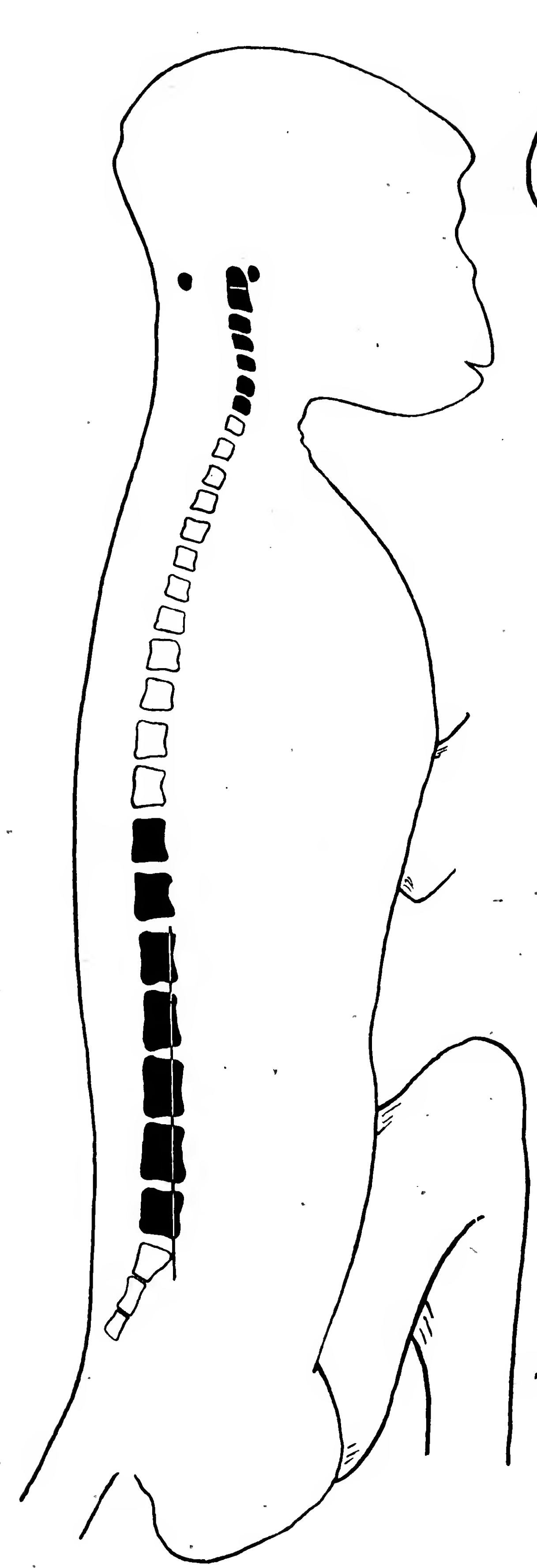


SIMIA SATYRUS
FEMALE

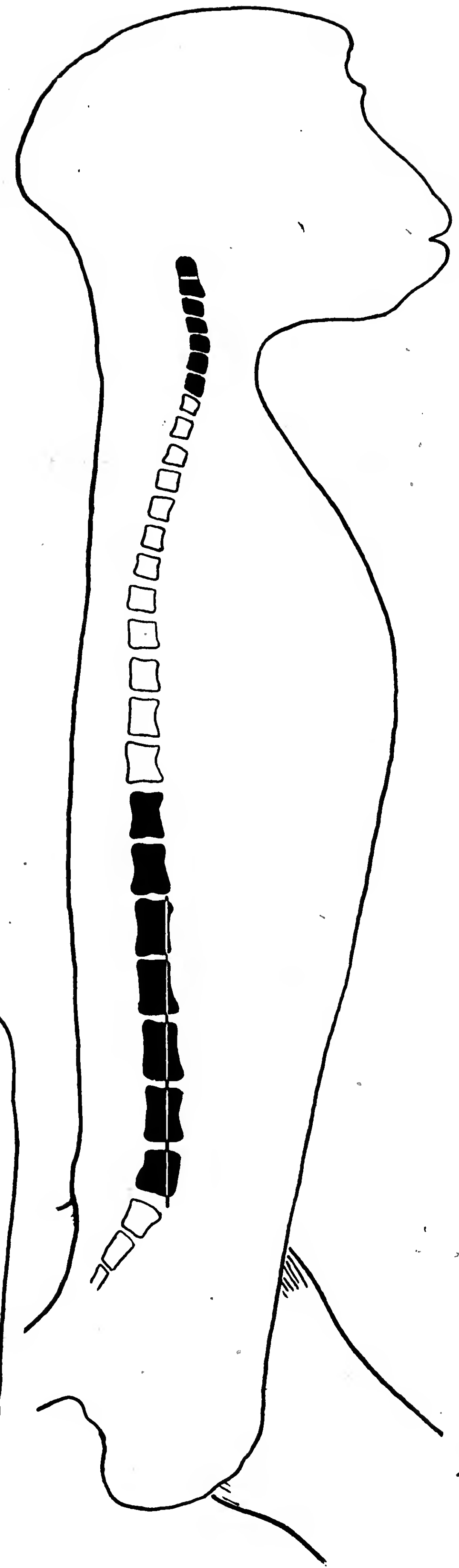


MACACUS NEMESTRINUS. No. 1 — MACACUS NEMESTRINUS. No. 2 — MACACUS RHESUS. — CERCOPITHECUS CAMPBELLI. No. 1 — CERCOPITHECUS CAMPBELLI. No. 2 — SEMNOPITHECUS ENTELLUS. — COLOBUS VELLEROSUS. No. 1 — COLOBUS VELLEROSUS. No. 2

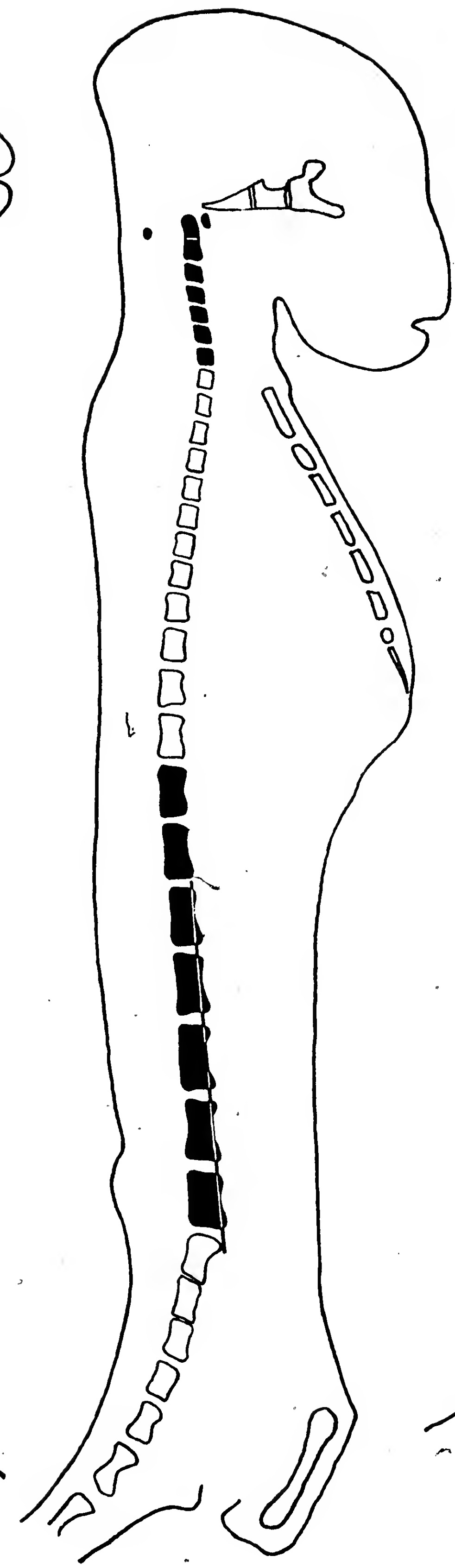
D. J. C. *del.*



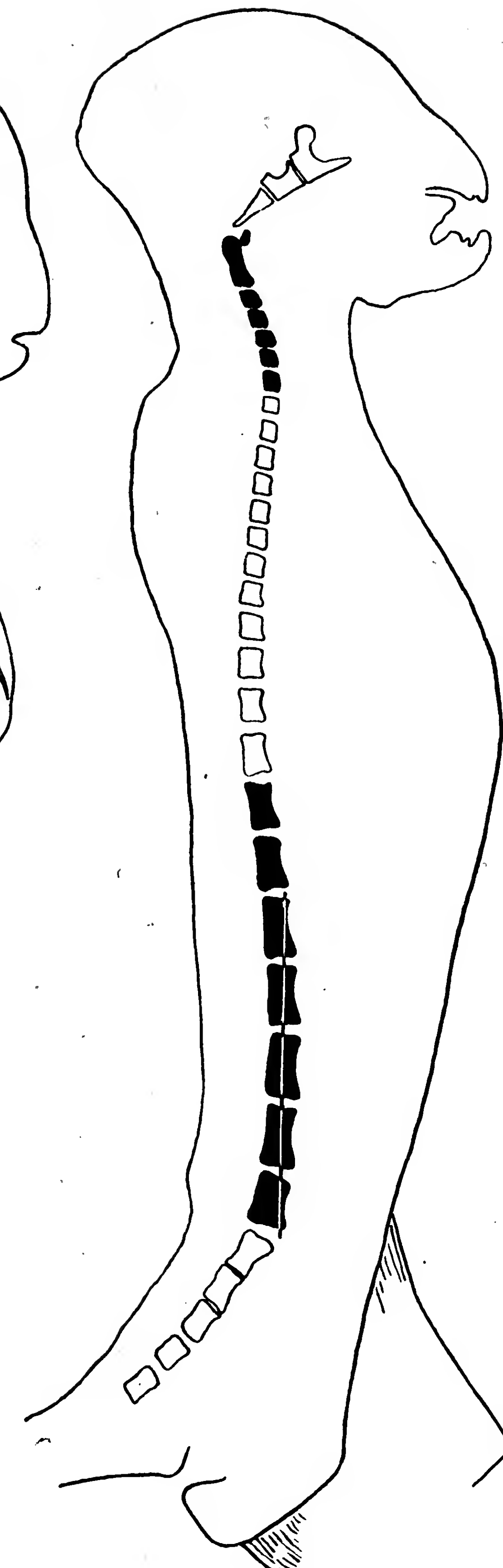
CERCOPITHECUS RUBER.
No. 1



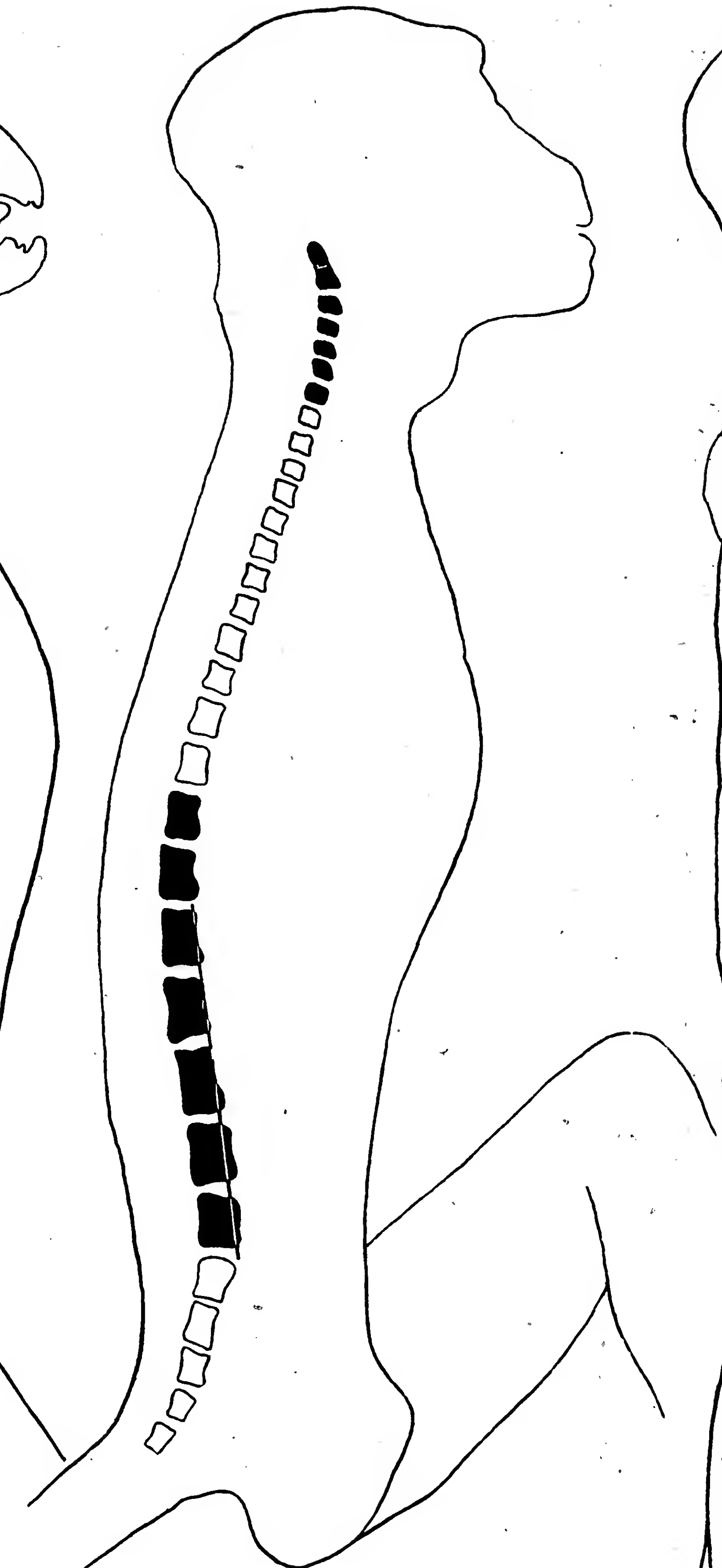
CERCOPITHECUS RUBER.
No. 2



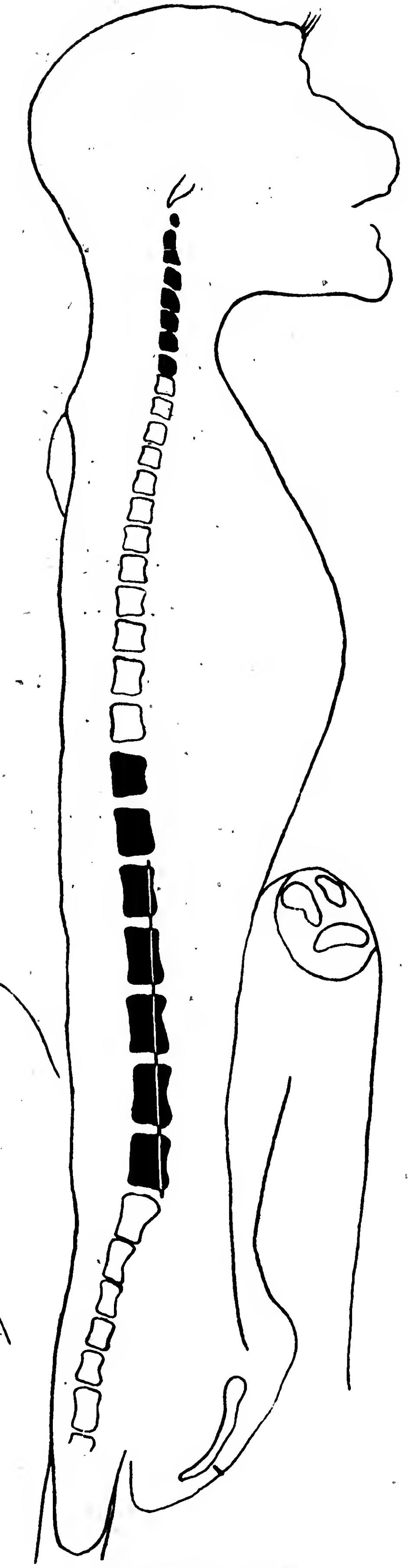
CERCOPITHECUS MONA.
No. 1



CERCOPITHECUS MONA.
No. 2

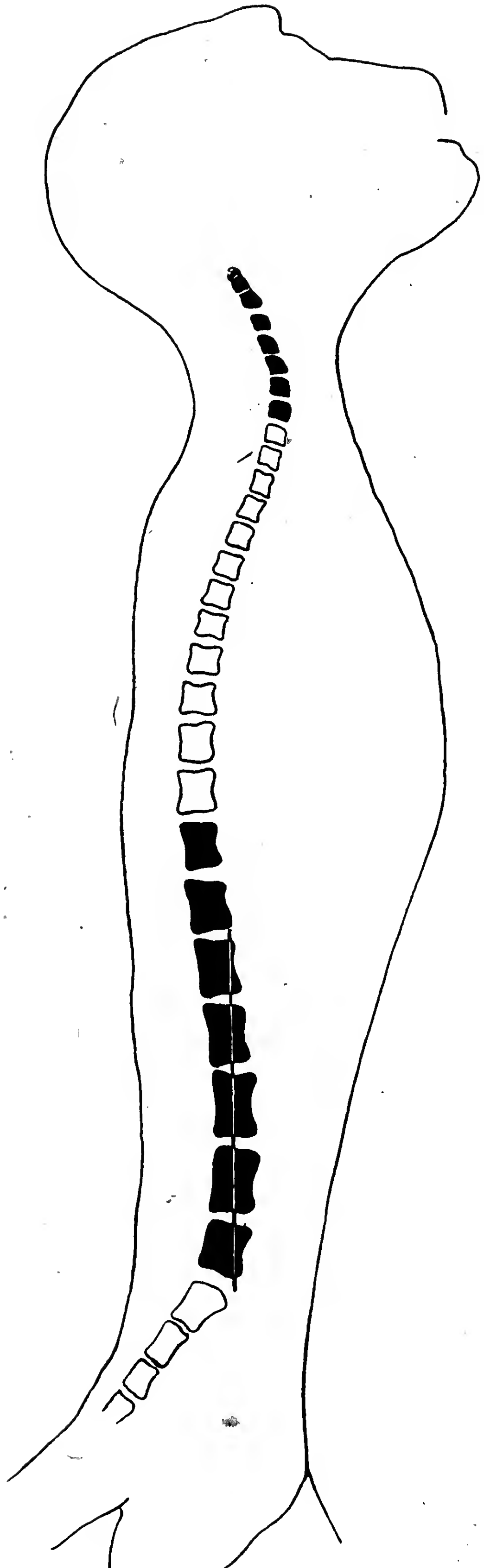


CERCOCEBUS FULIGINOSUS.
No. 1

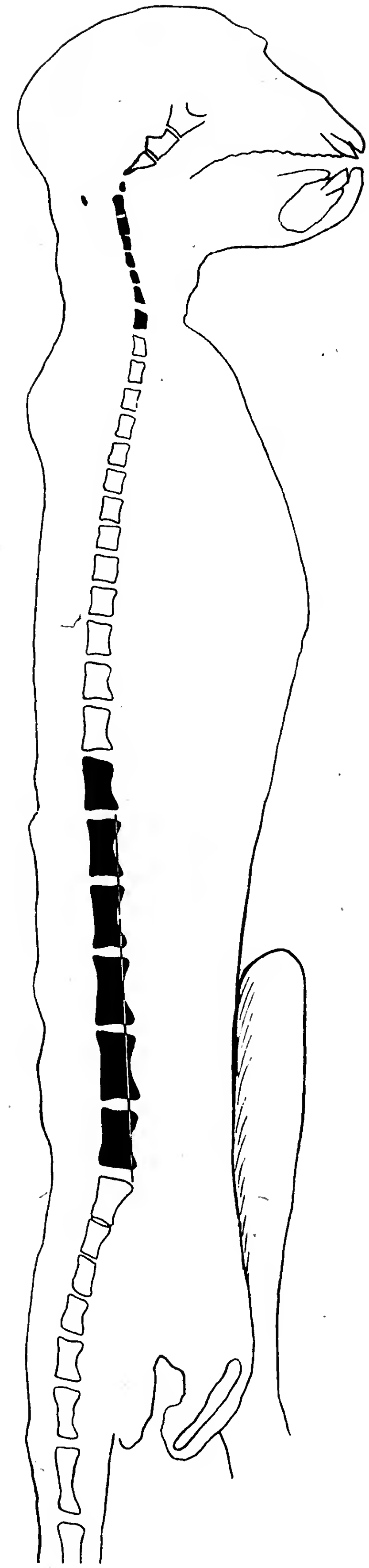


CERCOCEBUS FULIGINOSUS.
No. 2

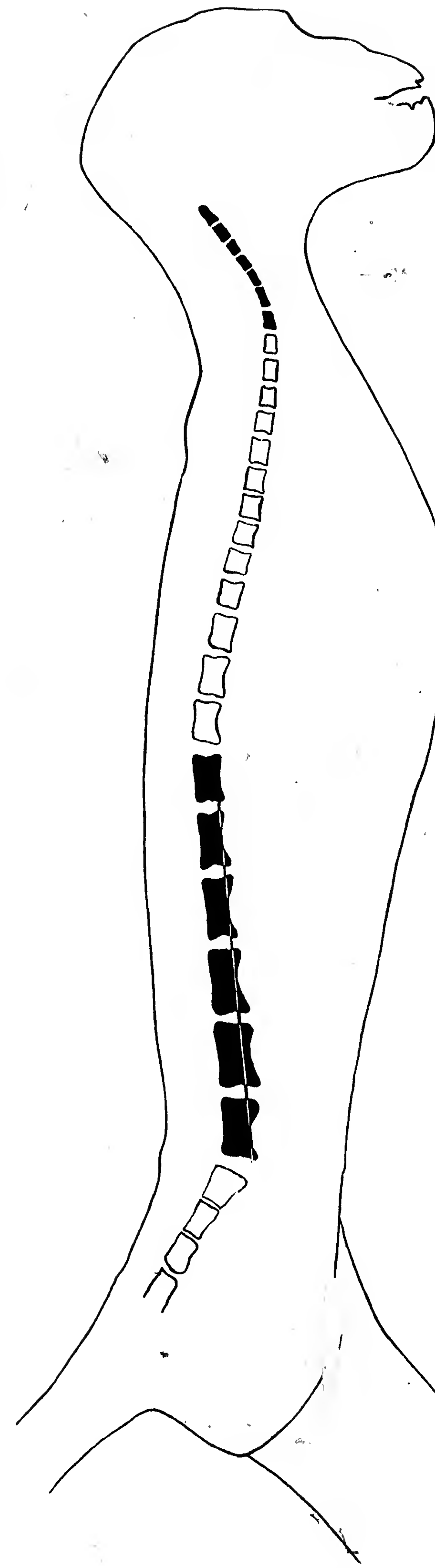
D. J. C. *del.*



Cercocebus fuliginosus.
No. 3

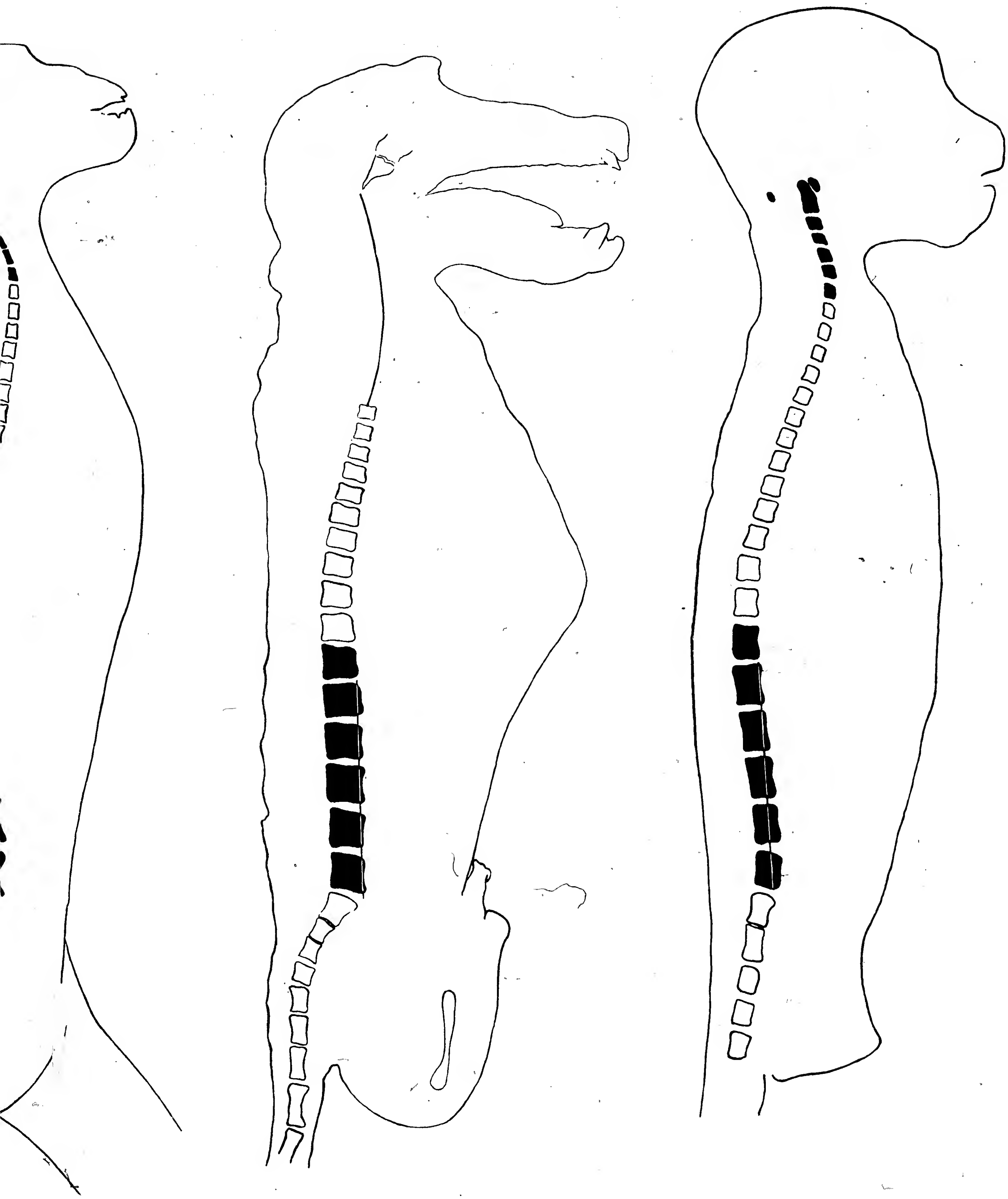


Chlorocebus sabaeus.
No. 1



Chlorocebus sabaeus.
No. 2

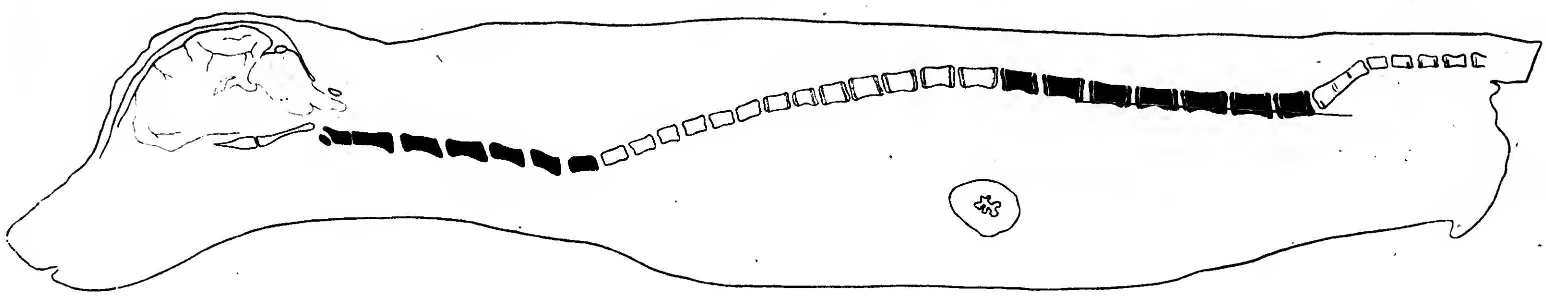
D. J. C. *del.*



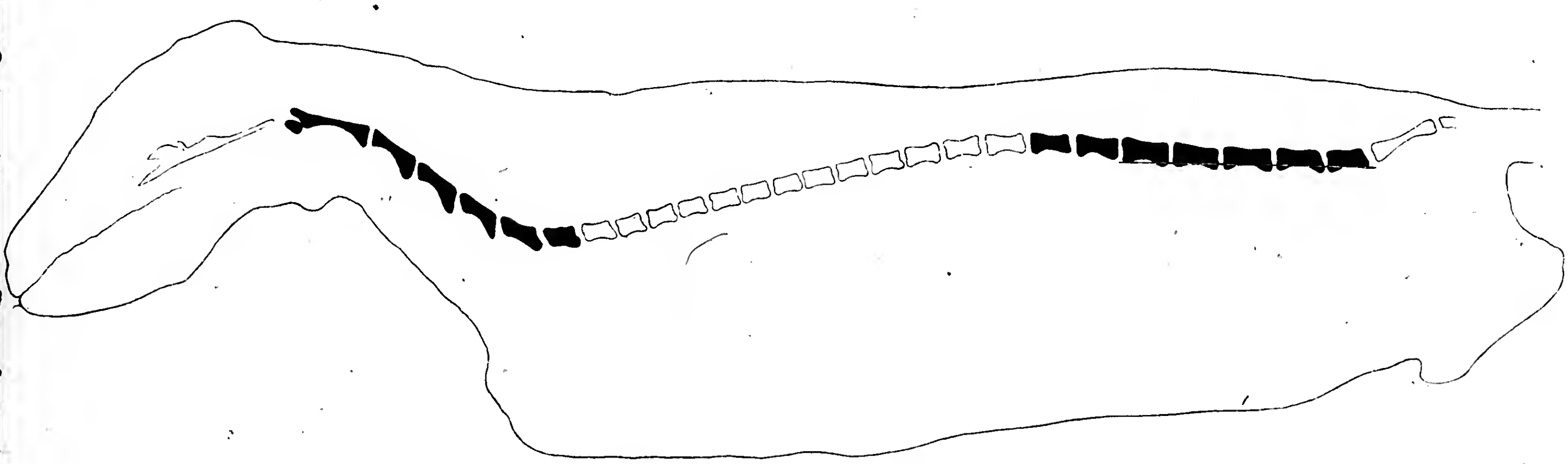
...BUS SABAEUS.
...o. 2

CYNOCEPALUS ANUBIS.

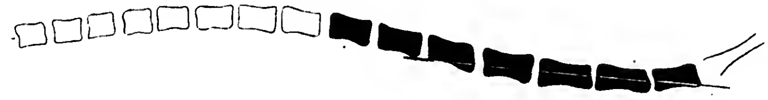
CEBUS CAPUCINUS.



Dog No. 1

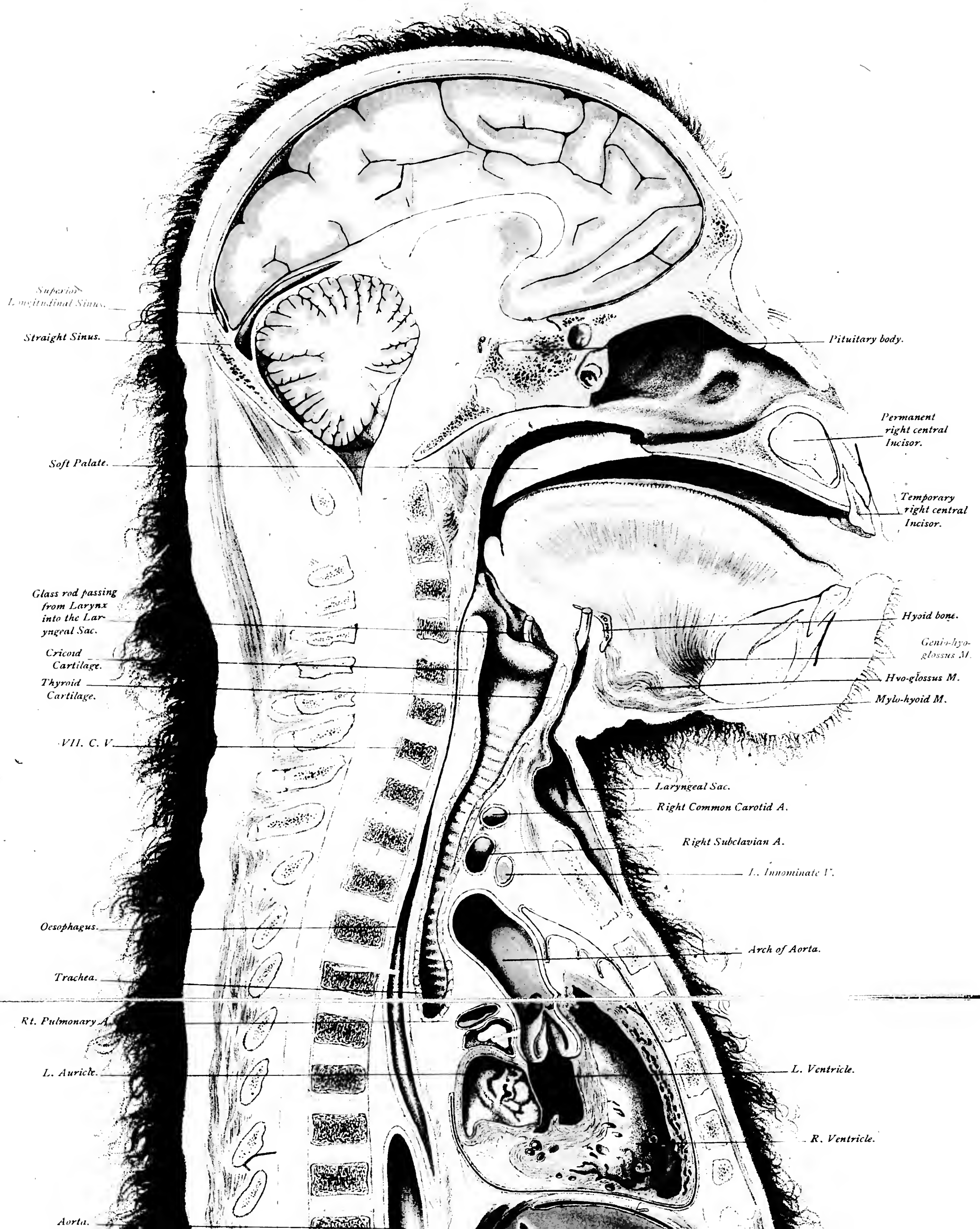


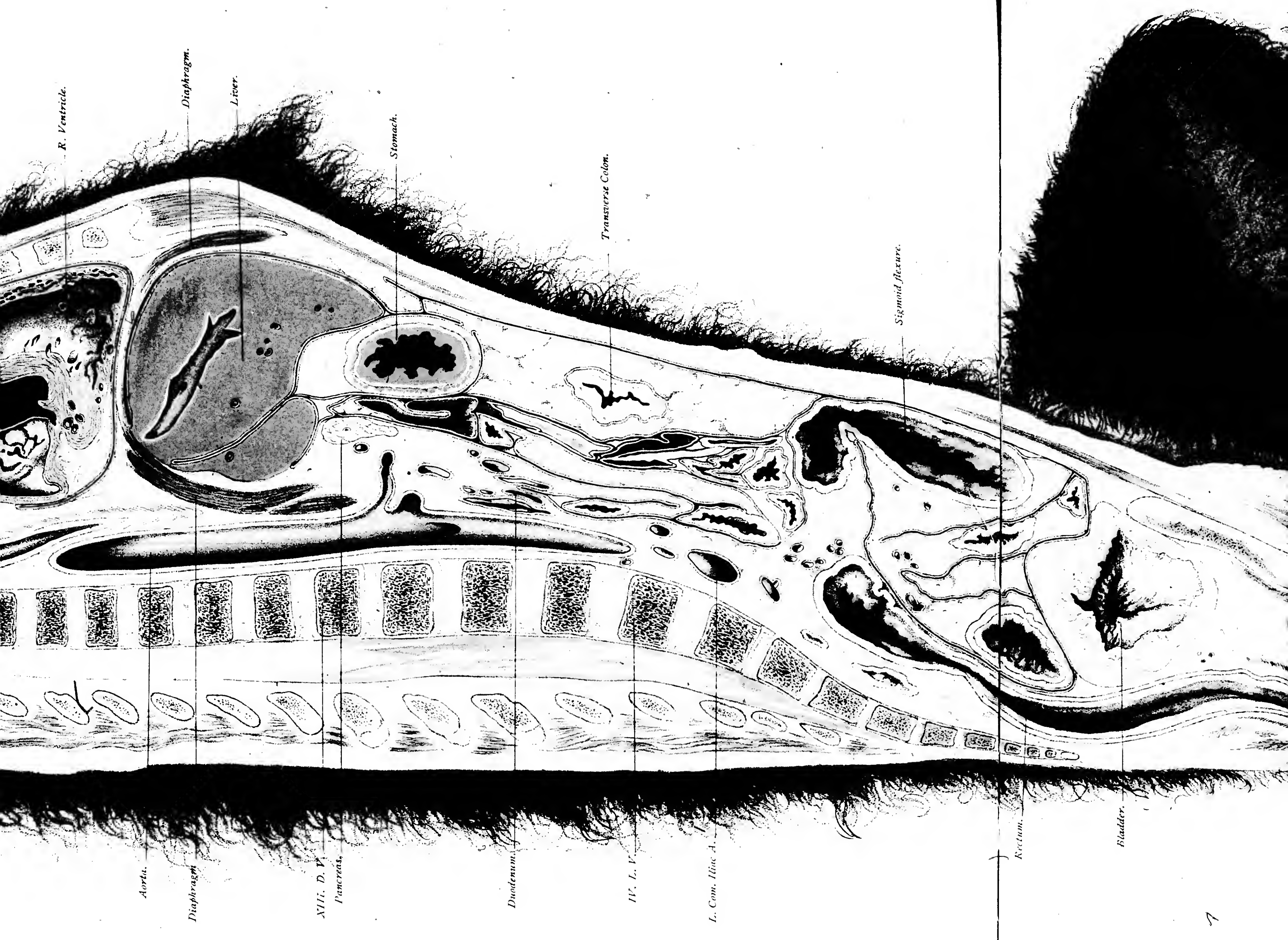
Dog No. 2

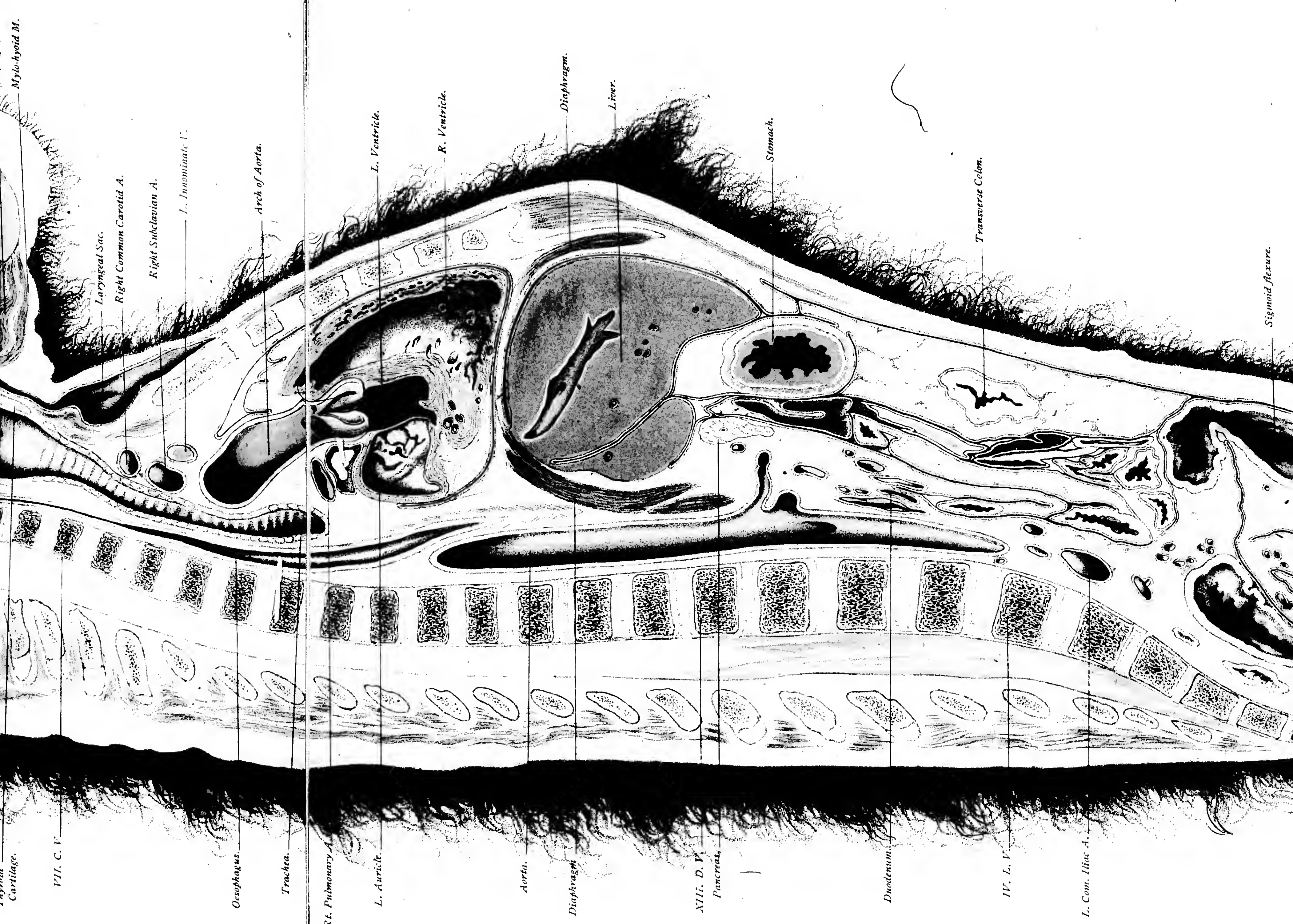


Dog No. 3

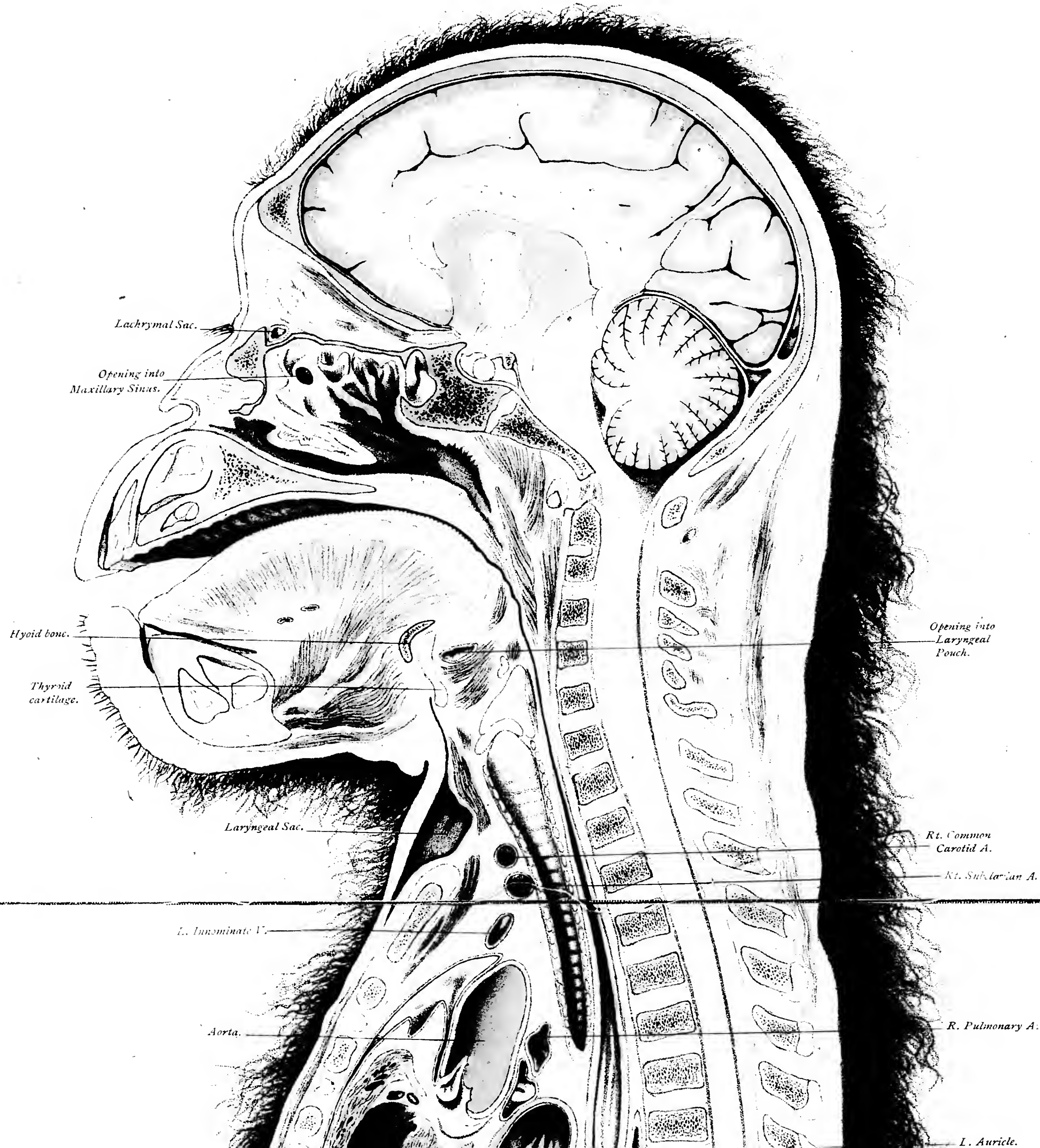
Bear

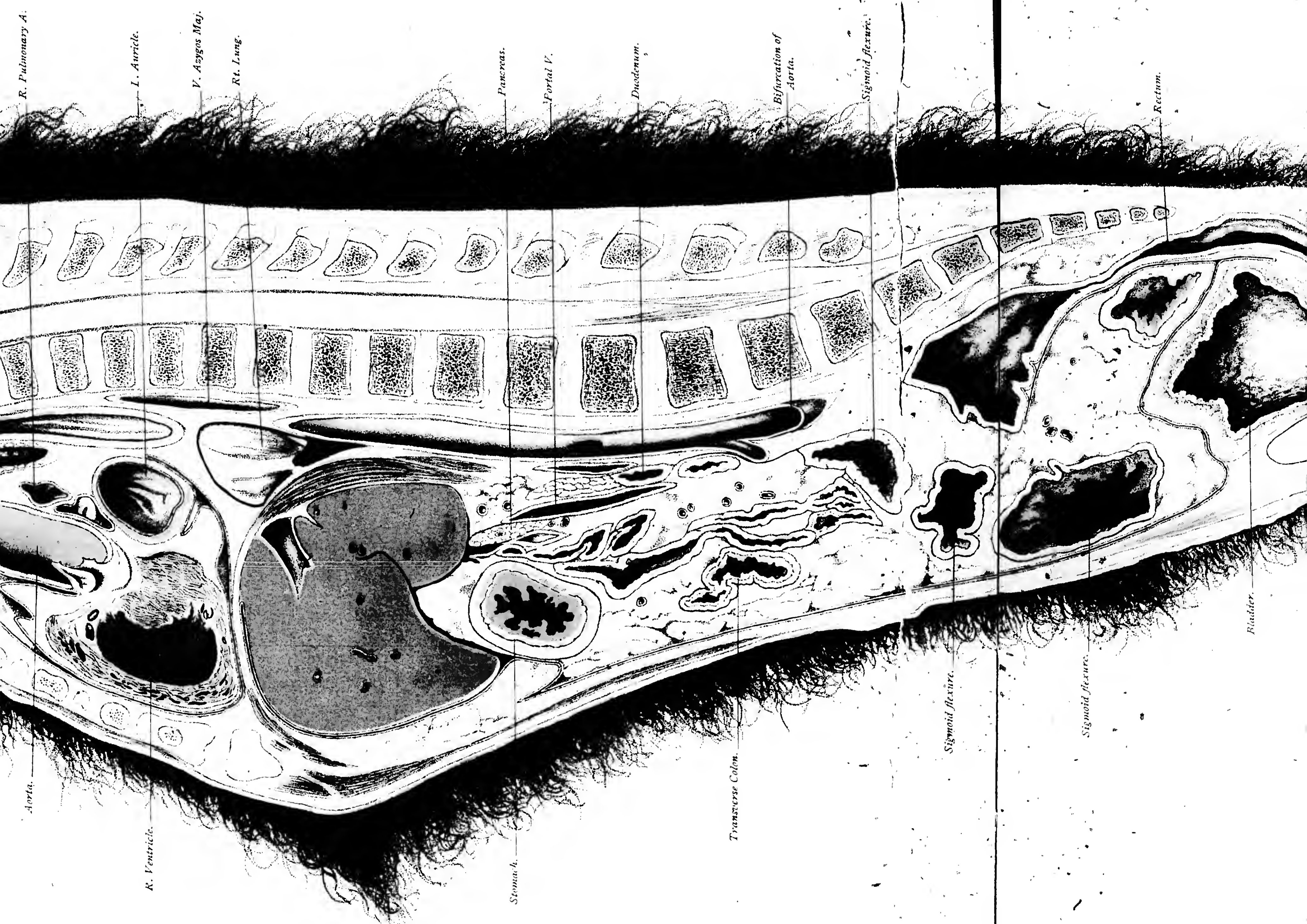

Superior Longitudinal Sinus.
Straight Sinus.
Soft Palate.
Glass rod passing from Larynx into the Laryngeal Sac.
Cricoid Cartilage.
Thyroid Cartilage.
VII. C. V.
Oesophagus.
Trachea.
Rt. Pulmonary A.
L. Auricle.
Aorta.
Pituitary body.
Permanent right central Incisor.
Temporary right central Incisor.
Hyoid bone.
Genio-hyo-glossus M.
Hyo-glossus M.
Mylo-hyoid M.
Laryngeal Sac.
Right Common Carotid A.
Right Subclavian A.
L. Innominate V.
Arch of Aorta.
L. Ventricle.
R. Ventricle.


R. Ventricle.
Diaphragm.
Liver.
Stomach.
Transverse Colon.
Sigmoid flexure.
Aorta.
Diaphragm.
XIII. D. V.
Pancreas.
Duodenum.
IV. L. V.
L. Com. Iliac A.
Rectum.
Bladder.

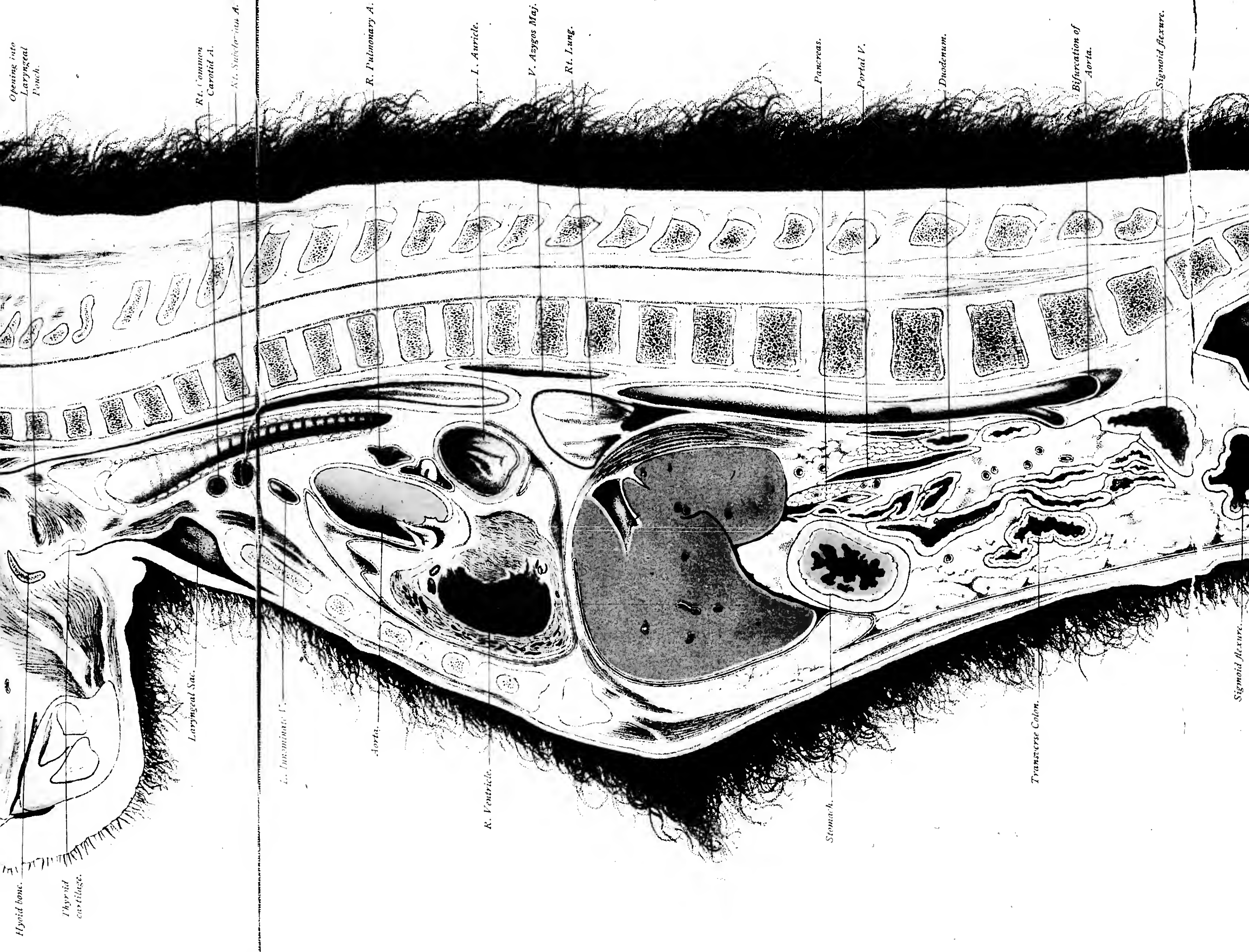

Mylo-hyoid M.
Laryngeal Sac.
Right Common Carotid A.
Right Subclavian A.
L. Innominate V.
Arch of Aorta.
L. Ventricle.
R. Ventricle.
Diaphragm.
Liver.
Stomach.
Transverse Colon.
Sigmoid flexure.
Cartilage.
VII. C. V.
Oesophagus.
Trachea.
Rt. Pulmonary A.
L. Auricle.
Aorta.
Diaphragm.
XIII. D. V.
Pancreas.
Duodenum.
IV. L. V.
L. Com. Iliac A.

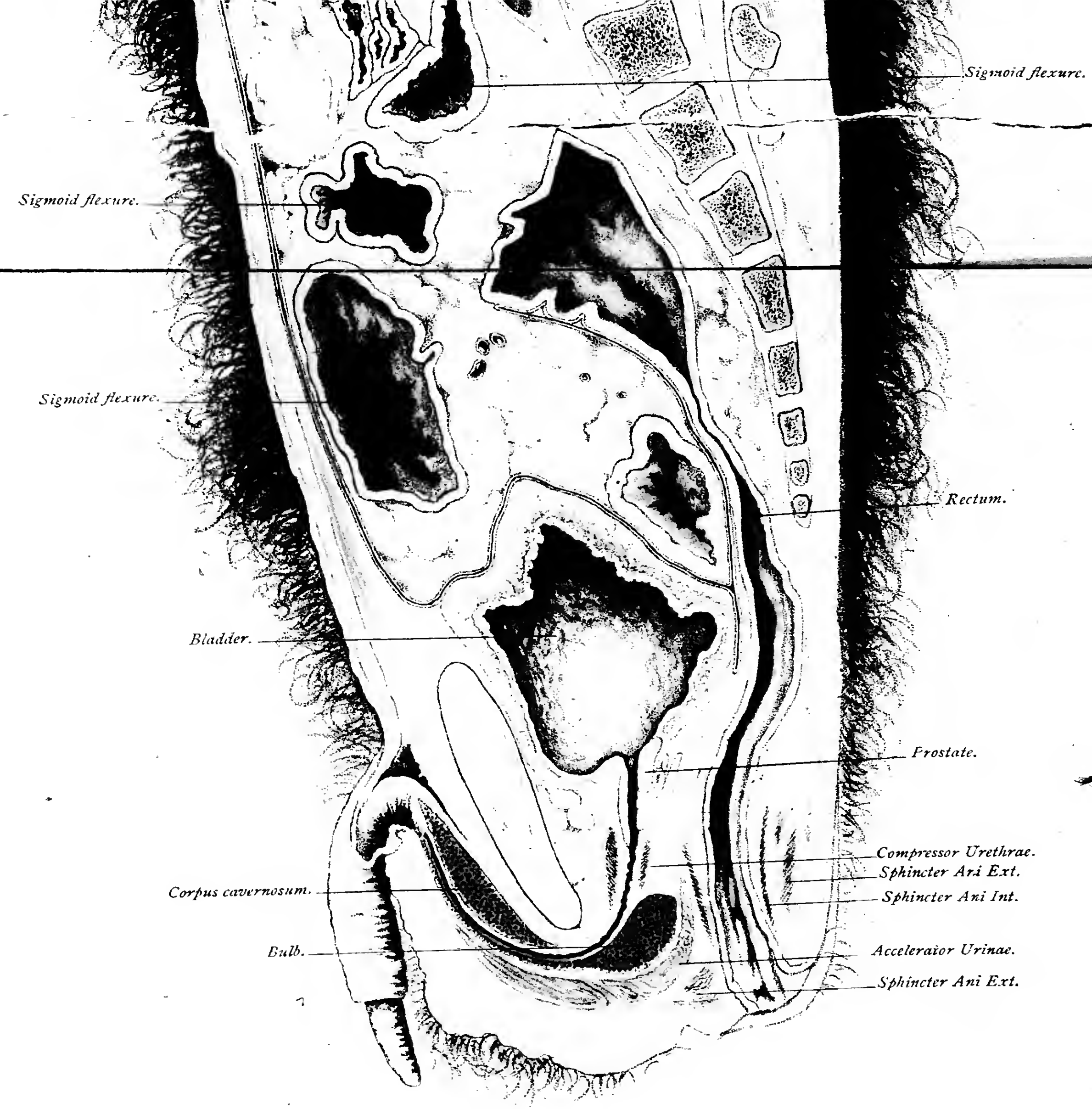
D. J. C., direx. J. BAYNE, del. et lith.


Lachrymal Sac.
Opening into Maxillary Sinus.
Hyoid bone.
Thyroid cartilage.
Laryngeal Sac.
L. Innominate V.
Aorta.
Opening into Laryngeal Pouch.
Rt. Common Carotid A.
Rt. Subclavian A.
R. Pulmonary A.
L. Auricle.



R. Pulmonary A.
L. Auricle.
V. Azygos Maj.
Rt. Lung.
Pancreas.
Portal V.
Duodenum.
Bifurcation of Aorta.
Sigmoid flexure.
Rectum.
Aorta.
R. Ventricle.
Stomach.
Transverse Colon.
Sigmoid flexure.
Sigmoid flexure.
Bladder.


Opening into Laryngeal Pouch.
Rt. Common Carotid A.
Rt. Subclavian A.
R. Pulmonary A.
L. Auricle.
V. Azygos Maj.
Rt. Lung.
Pancreas.
Portal V.
Duodenum.
Bifurcation of Aorta.
Sigmoid flexure.
Hyoid bone.
Thyroid cartilage.
Laryngeal Sac.
L. Innominate V.
Aorta.
R. Ventricle.
Stomach.
Transverse Colon.
Sigmoid flexure.



D. J. C., direx. J. BAYNE, del. et lith.

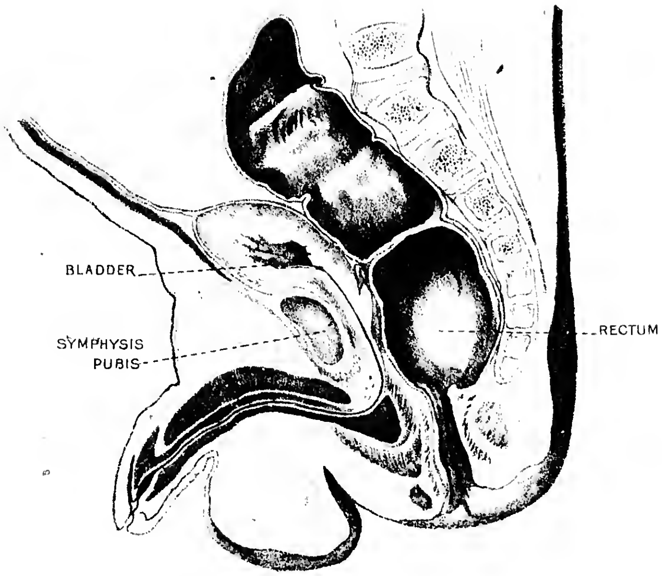


Fig. 1.

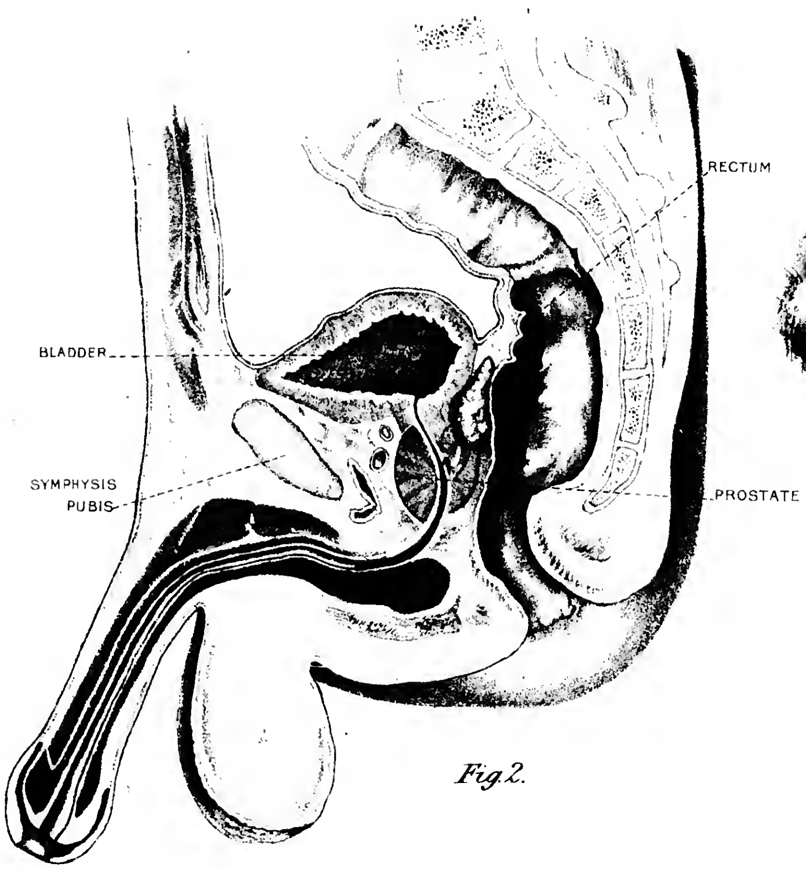


Fig. 2.

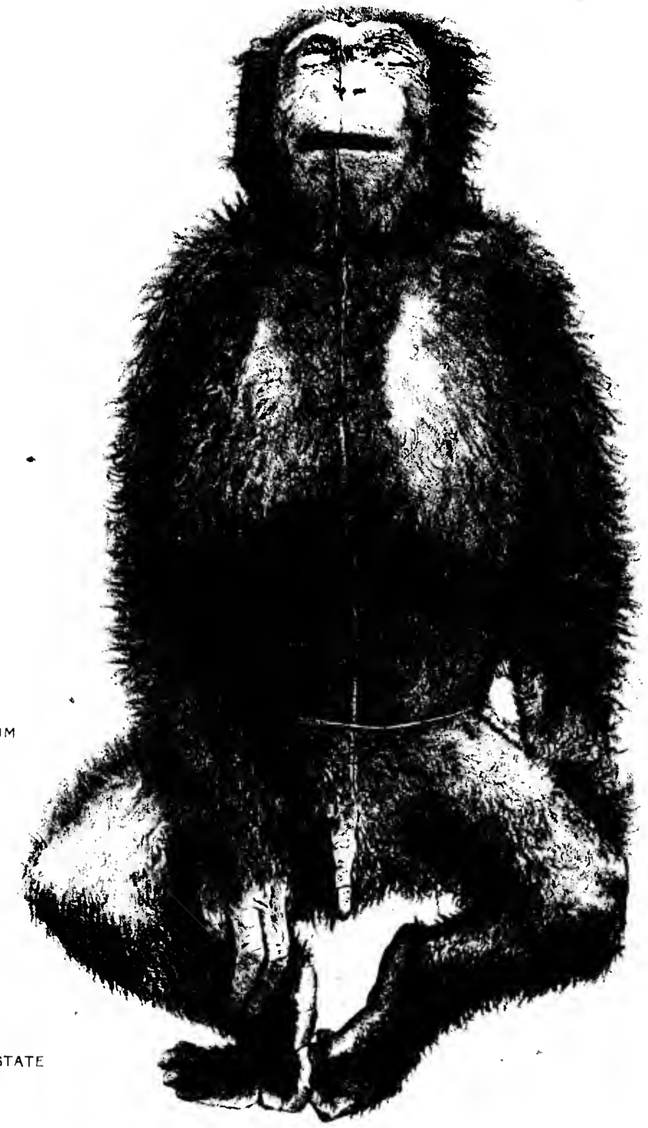
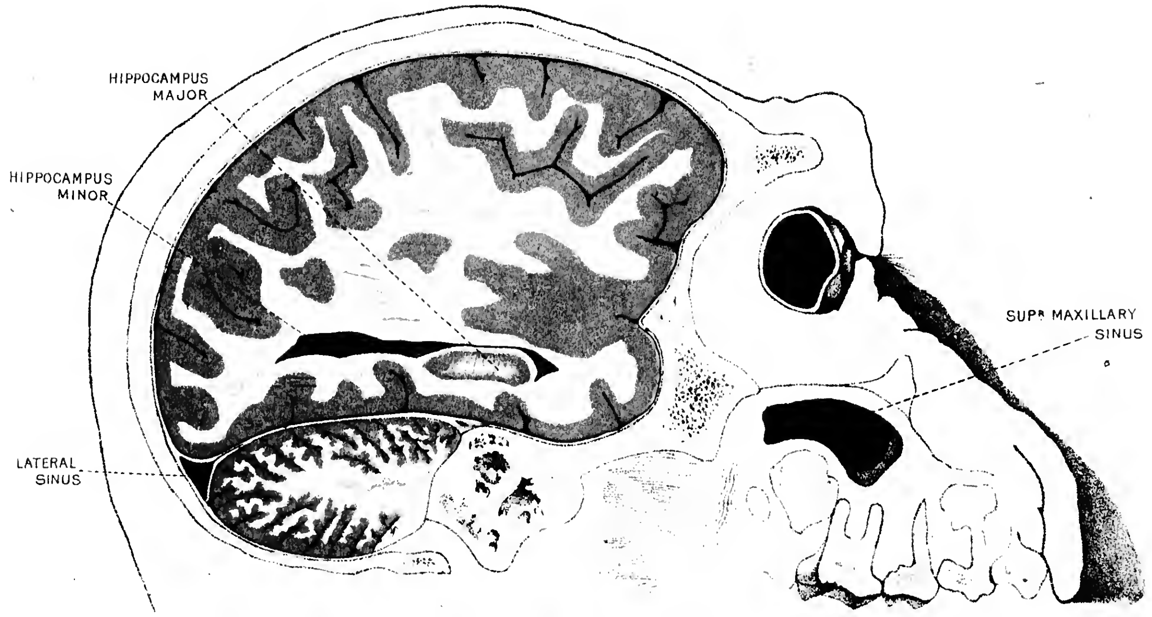


Fig. 3.

BAYNE, *del. et lith.* BANKS & Co EDINBURGH



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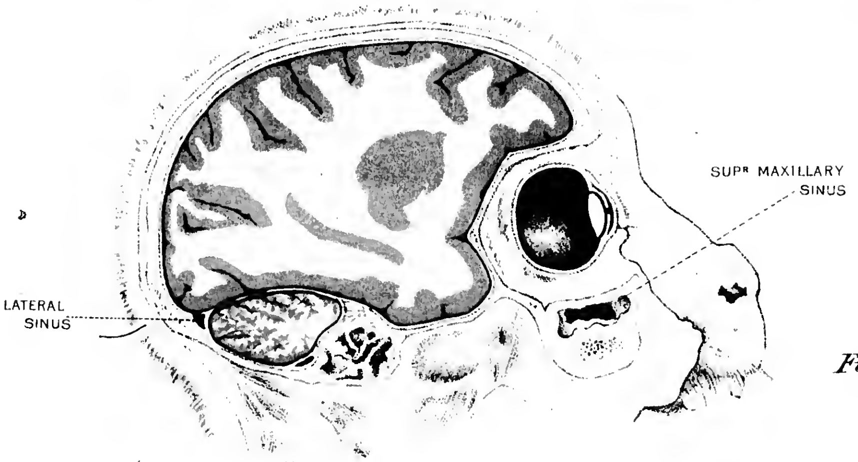


Fig. 2.

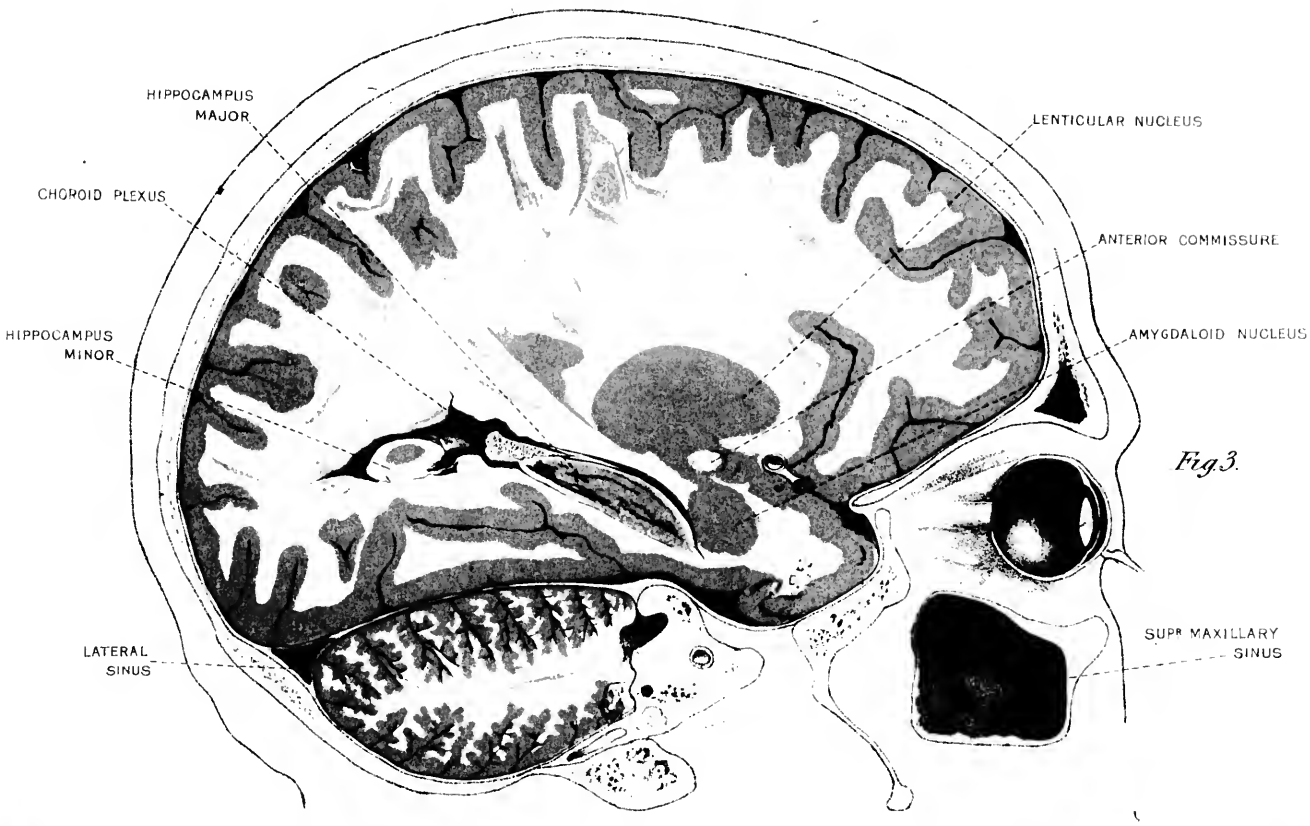


Fig. 3.

BAYNE, *del. et lith.*

BANKS & Co EDINBURGH

Fig. 2.

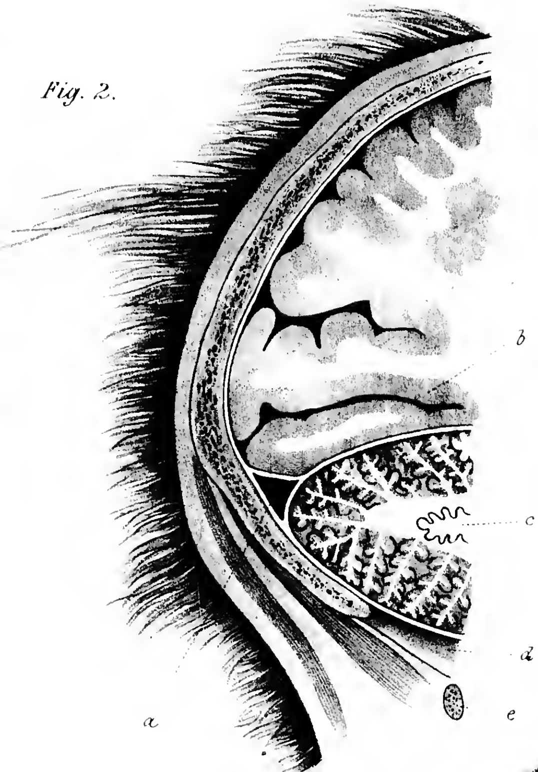


Fig. 3.

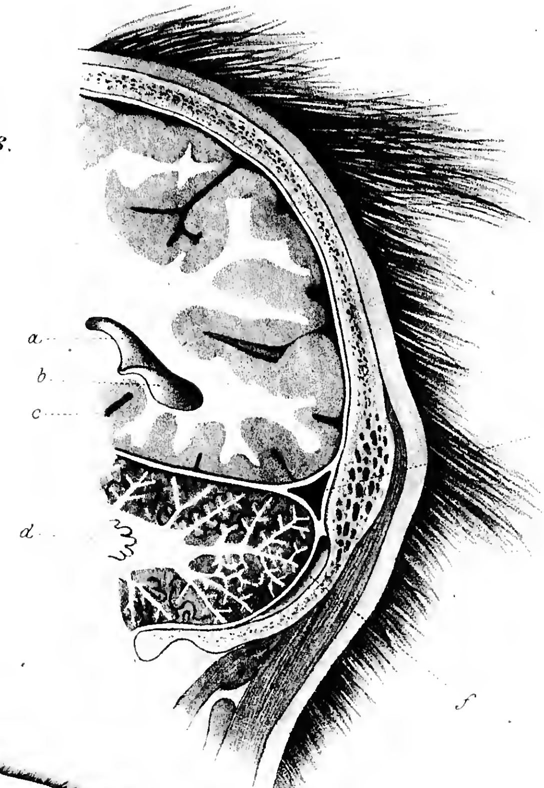
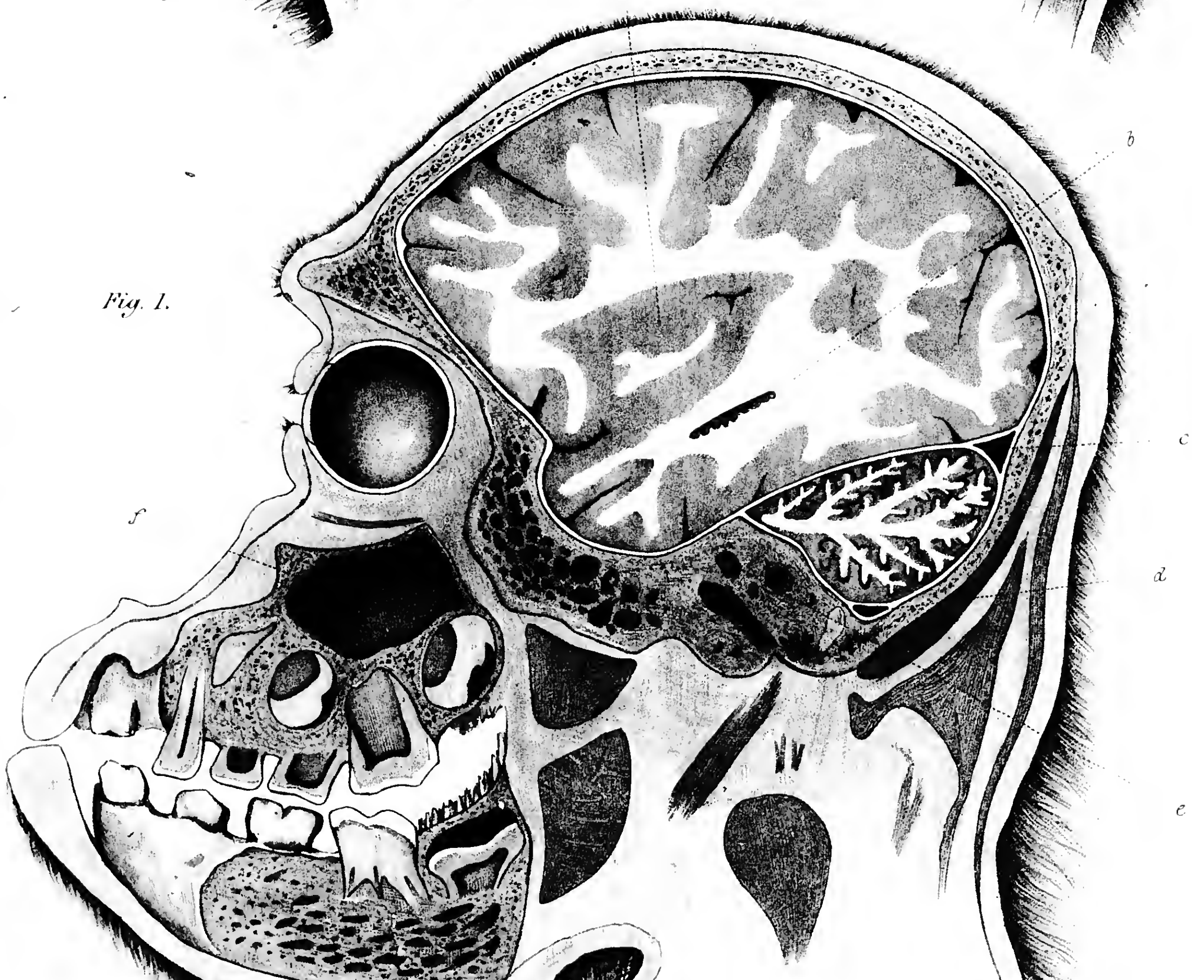


Fig. 1.



H. St John Brooks M.B. del.

F. Huth, Lithr Edinr

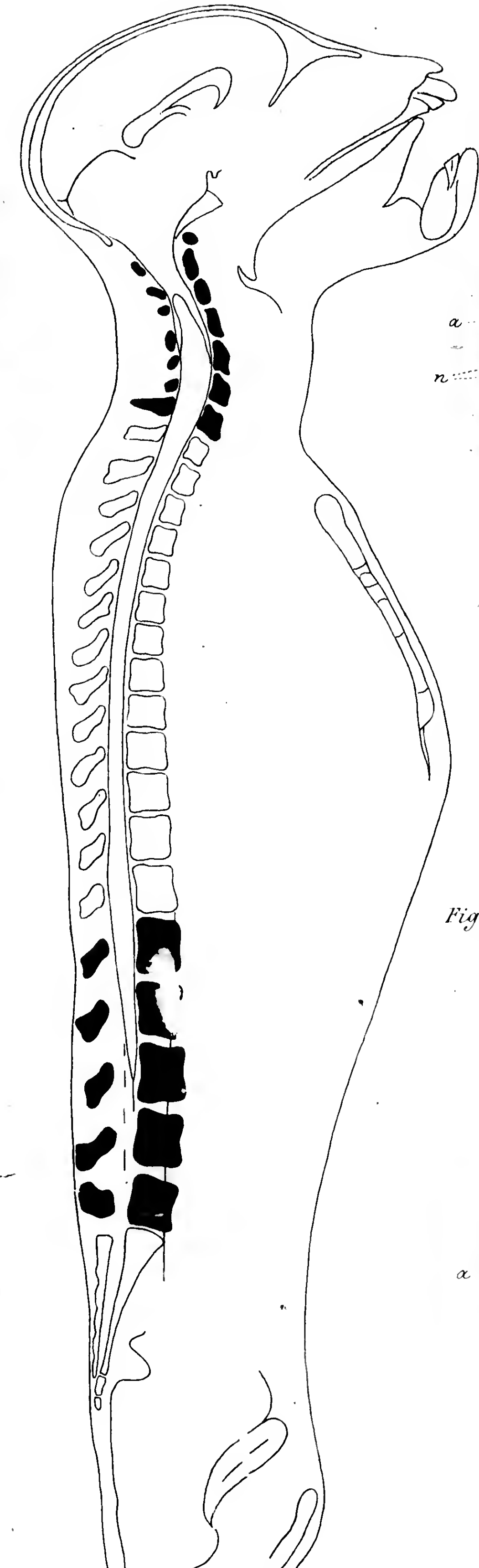


Fig. 1.

HYLOBATES AGILIS
(WAUWAU) ♀

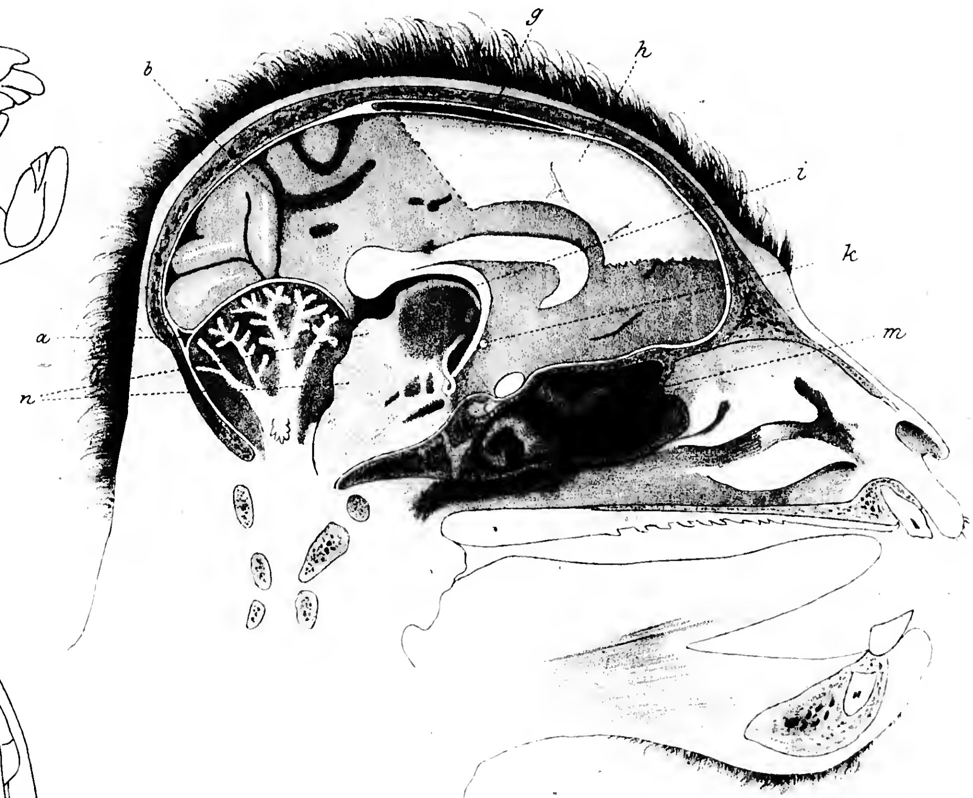


Fig. 2.

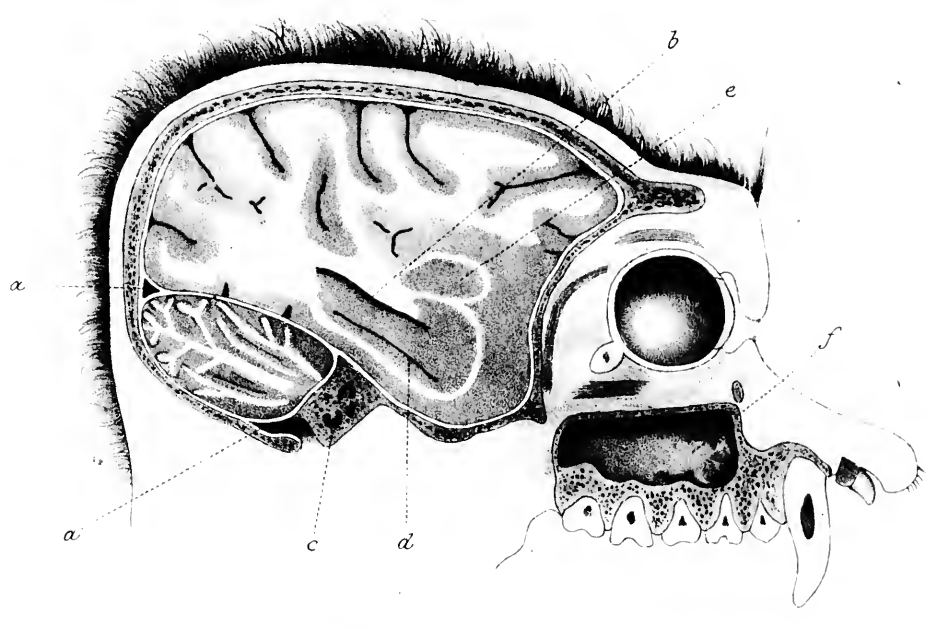


Fig. 3.

H St. John Brooks M.B. del. F Huth, Lithr Edinr

ROYAL IRISH ACADEMY.

"CUNNINGHAM MEMOIRS."

No. III.

NEW RESEARCHES

ON

SUNHEAT, TERRESTRIAL RADIATION,

ETC.

ROYAL IRISH ACADEMY.

"CUNNINGHAM MEMOIRS."

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BY

REV. SAMUEL HAUGHTON, M.A., M.D., Dubl.;
D.C.L., Oxon.; LL.D., Cantab. & Edin.
(Fellow of Trinity College, Dublin.)

With Nine Plates.

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"CUNNINGHAM MEMOIRS."

New Researches on Sunheat, Terrestrial Radiation, &c. By the Rev. Samuel Haughton, M.D., Dubl.; D.C.L., Oxon.; LL.D., Cantab. & Edin. (With Plates 1 to 9.)

[Read, November 9, 1885].

PART III.*

SUNHEAT ACTUALLY RECEIVED, MONTH BY MONTH, AT GREENWICH, DEDUCED FROM THE HOURLY OBSERVATIONS OF THE THERMOMETER.†

1.—Month of January.

Table I.—*Mean Hourly Observations of Temperature.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	37°·9 F.	12	Noon.	41°·2 F.
1	1 A.M.	38 ·0 ,,	13	1 P.M.	41 ·8 ,,
2	2 ,,	37 ·7 ,,	14	2 ,,	41 ·9 ,,
3	3 ,,	37 ·6 ,,	15	3 ,,	41 ·4 ,,
4	4 ,,	37 ·3 ,,	16	4 ,,	40 ·8 ,,
5	5 ,,	37 ·1 ,,	17	5 ,,	40 ·0 ,,
6	6 ,,	37 ·0 ,,	18	6 ,,	39 ·5 ,,
7	7 ,,	37 ·0 ,,	19	7 ,,	39 ·2 ,,
8	8 ,,	37 ·4 ,,	20	8 ,,	38 ·8 ,,
9	9 ,,	37 ·9 ,,	21	9 ,,	38 ·5 ,,
10	10 ,,	38 ·7 ,,	22	10 ,,	38 ·3 ,,
11	11 ,,	40 ·2 ,,	23	11 ,,	38 ·2 ,,

This Table is plotted to scale in Plate 1.

* For Parts I. and II., *vide* "Transactions, Royal Irish Academy," Vol. xxviii., Pt. 6, 1881.
† During a period of thirty-five years.

From sunset to sunrise we find—

TABLE II.—*Nocturnal Radiation* (January).

Time.	Temperature.	Hourly Diminution.
4 P.M.—5 P.M.	40°·40 F.	− 0°·80 F.
5 ,, 6 ,,	39 ·75 ,,	− 0 ·50 ,,
6 ,, 8 ,,	39 ·17 ,,	− 0 ·35 ,,
8 ,, 10 ,,	38 ·53 ,,	− 0 ·25 ,,
10 ,, midnight.	38 ·13 ,,	− 0 ·20 ,,
Midnight to 7 A.M.	37 ·45 ,,	− 0 ·12 ,,

This Table determines the Radiation from 37°·45 F. to 40°·4 F., and it may be extended to higher temperatures, by means of the sunlight observations, as follows:—

TABLE III.—*From Sunrise to Sunset* (January).

Hour.	θ Temperature.	$\frac{d\theta}{dt}$ Hourly Change of Temperature.	cos z.*
8 A.M.— 9 A.M.	37°·65 F.	+ 0·5	0·072
9 ,, —10 ,,	38 ·30 ,,	+ 0·8	0·179
10 ,, —11 ,,	39 ·45 ,,	+ 1·5	0·255
11 ,, —Noon.	40 ·70 ,,	+ 1·0	0·294
Noon — 1 P.M.	41 ·50 ,,	+ 0·6	0·294
1 P.M.— 2 ,,	41 ·85 ,,	+ 0·1	0·255
2 ,, — 3 ,,	41 ·65 ,,	− 0·5	0·179
3 ,, — 4 ,,	41 ·10 ,,	− 0·6	0·072

* $\cos z = \sin\lambda \sin\delta + \cos\lambda \cos\delta \cos h$.

This becomes, for Greenwich and 15th January—

$$\cos z = -\ 0{\cdot}282 + 0{\cdot}581 \cos h.$$

Sunrise $= 8^{h}\ 10^{m}$ A.M.
Sunset $= 4^{h}\ 8^{m}$ P.M.

If S and R denote the hourly effect of Sunheat and Radiation, we have

$$\frac{d\theta}{dt} = S + R.$$

Let $S_1 \ldots . S_4$ denote the hourly Sunheat in the forenoon and afternoon, and $R_1 \ldots . R_8$ the Radiation for each of the eight hours; then we have the following eight equations:—

$$\begin{aligned} S_1 + R_1 &= +\ 0{\cdot}5, \\ S_2 + R_2 &= +\ 0{\cdot}8, \\ S_3 + R_3 &= +\ 1{\cdot}5, \\ S_4 + R_4 &= +\ 1{\cdot}0, \\ S_4 + R_5 &= +\ 0{\cdot}6, \\ S_3 + R_6 &= +\ 0{\cdot}1, \\ S_2 + R_7 &= -\ 0{\cdot}5, \\ S_1 + R_8 &= -\ 0{\cdot}6. \end{aligned}$$

Here we have eight equations and twelve unknowns; but $R_1 \ldots . R_4$ are known from Table II., and therefore the remaining eight can be found—

$$\begin{aligned} R_{(37{\cdot}65)} &= -\ 0{\cdot}14, \\ R_{(38{\cdot}30)} &= -\ 0{\cdot}21, \\ R_{(39{\cdot}45)} &= -\ 0{\cdot}42, \\ R_{(40{\cdot}70)} &= -\ 0{\cdot}95. \end{aligned}$$

Substituting, we find,

$$\begin{aligned} S_1 &= +\ 0{\cdot}64, \\ S_2 &= +\ 1{\cdot}01, \\ S_3 &= +\ 1{\cdot}92, \\ S_4 &= +\ 1{\cdot}95; \end{aligned}$$

and, substituting these values of Sunheat in the afternoon observations, we have

$$R_{(41\cdot50)} = -1\cdot35,$$
$$R_{(41\cdot85)} = -1\cdot82,$$
$$R_{(41\cdot65)} = -1\cdot51,$$
$$R_{(41\cdot10)} = -1\cdot24.$$

Our complete Table of Radiation by night and day thus becomes—

TABLE IV.—*Radiation* (January).

θ.	R.		θ.	R.	
37°·45 F.	− 0·13	Night.	41°·10 F.	− 1·24	Day.
38 ·13 ,,	− 0·20		41 ·50 ,,	− 1·35	
38 ·53 ,,	− 0·25		41 ·65 ,,	− 1·51	
39 ·17 ,,	− 0·35		41 ·85 ,,	− 1·82	
39 ·75 ,,	− 0·50				
40 ·40 ,,	− 0·80				

If the foregoing Table be plotted, and read off with equal differences of Radiation, we find the following :—

TABLE IV. (*bis*).—*Radiation* (January).

	θ.	$\frac{d\theta}{dt}$.	$\theta - \theta_0$.*
Night,	38·13	− 0·20	1·13
	39·37	− 0·40	2·37
	39·97	− 0·60	2·97
	40·40	− 0·80	3·40
Day,	40·84	− 1·00	3·84
	41·37	− 1·20	4·37
	41·57	− 1·40	4·57
	41·72	− 1·60	4·72
	41·84	− 1·80	4·84

* This excess of temperature above the lowest temperature is now usually called the "range," by Meteorologists— [θ_0 = 37°·0 F.]

The third column contains the excess of temperature above the lowest temperature before sunrise when the Radiation appears to cease (37° F.). I have plotted it, in terms of the Radiation, in Plate 6.

At temperatures not much above θ_0 the Radiation is proportional to $\theta - \theta_0$, but at higher temperatures it increases faster than this difference.

I found, on consideration, that it may be best represented by a parabolic curve, having its axis vertical, and its vertex at the origin.

This curve would have the form

$$(\theta - \theta_0)^n = p\frac{d\theta}{dt};$$

or, taking the logarithms of both sides,

$$n \log(\theta - \theta_0) = \log(p) + \log\frac{d\theta}{dt}.$$

Using the last five observations of the preceding Table, we obtain five linear equations, from which I find the most probable values of n and p to be

$$n = 2{\cdot}04, \quad \log p = 1{\cdot}166, \quad p = 14{\cdot}66.$$

This is nearly the common parabola, and I have drawn a portion of it in Plate (6) from a to b, for the purpose of comparison with the observations. The night Radiation is represented by the right line

$$\frac{\theta - \theta_0}{d\theta} = 4{\cdot}25,$$

or

$$d\theta = -\,0{\cdot}235\,(\theta - \theta_0).$$

The day Radiation is best represented by the parabolic curve

$$(\theta - \theta_0)^{2\cdot04} = 14{\cdot}66\frac{d\theta}{dt}.$$

The Sunheat values may be assumed, approximately, proportional to the cosines of the Sun's zenith distance; or,

$$S_1 = 0{\cdot}072\,a_1 = +\,0{\cdot}64,$$
$$S_2 = 0{\cdot}179\,a_1 = +\,1{\cdot}01,$$
$$S_3 = 0{\cdot}255\,a_1 = +\,1{\cdot}92,$$
$$S_4 = 0{\cdot}294\,a_1 = +\,1{\cdot}95;$$

or,
$$a_1 = \begin{array}{r} 8{\cdot}89 \\ 5{\cdot}65 \\ 7{\cdot}53 \\ 6{\cdot}64 \\ \hline 7{\cdot}18 \end{array} \text{ per hour.}$$

This represents the hourly rise in temperature that would be produced by the January sun, if placed in the zenith.

If we examine either Table I. or its equivalent, Plate 1, we see that there are four points of importance—

(1) Where $\dfrac{d\theta}{dt} = 0$;

(2) Where $\dfrac{d^2\theta}{dt^2} = 0$.

(1) The first pair of points give

$$(a)\quad S + R = \frac{d\theta}{dt} = 0;$$

$$(b)\quad \theta - \theta_0 = 0.$$

These points are the diurnal maximum and minimum of the temperatures, and give

$$\theta - \theta_0 = 0,$$

and
$$a \cos z = \frac{(\theta - \theta_0)^n}{p}.$$

(2) The other pair of singular points on the curve where $\dfrac{d^2\theta}{dt^2} = 0$, or $\dfrac{d\theta}{dt}$ = maximum, or minumum give the following:

$$\frac{d\theta}{dt} = S + R,$$

$$\frac{d^2\theta}{dt^2} = \frac{dS}{dt} + \frac{dR}{d\theta}(S + R) = 0.$$

The best mode of discussing these points is to construct the curve of hourly change of temperature, in terms of the time. This is done in Plate 3, in which $\frac{d\theta}{dt} = 0$ corresponds to the points at which the curve crosses the axis of x, and $\frac{d^2\theta}{dt^2} = 0$ corresponds to the points at which the ordinate is a maximum or minimum.

We thus find

$$(a) \quad \frac{d\theta}{dt} = 0 \left\langle \begin{array}{lll} 6^h\ 30^m, \text{ A.M.}, & \theta = 37°{\cdot}0. & \\ 1^h\ 44^m, \text{ P.M.}, & \theta = 41°{\cdot}90, & h = 26°. \end{array} \right.$$

$$(b) \quad \frac{d^2\theta}{dt^2} = 0 \left\langle \begin{array}{lll} 10^h\ 24^m, \text{ A.M.}, & \theta = 39°{\cdot}3, & h = 24°. \\ 4^h\ 00^m, \text{ P.M.}, & \theta = 40°{\cdot}8, & h = 60°. \end{array} \right.$$

Hence we find

$$(a) \quad \left\{ \begin{array}{l} \theta_1 - \theta_0 = 0, \\ ap \cos z = (\theta - \theta_0)^n. \end{array} \right.$$

$$(b) \quad \frac{ap\beta}{n} \sin h = (S + R)(\theta - \theta_0)^{n-1}.$$

From the given data, we find

(1) $\theta_0 = 37°{\cdot}0$;

(2) $7{\cdot}18 \times 14{\cdot}66 \times 0{\cdot}240 = (4{\cdot}90)^n$, or $n = 2{\cdot}032$;

(3) $\frac{ap\beta}{n} \times 0{\cdot}407 = 1{\cdot}51 \times (2{\cdot}3)^{n-1}$, $S + R = 1{\cdot}51$;

(4) $\frac{ap\beta}{n} \times 0{\cdot}866 = 0{\cdot}85 \times (3{\cdot}8)^{n-1}$, $S + R = 0{\cdot}85$.

Dividing (3) by (4), so as to eliminate all the unknowns, except n, we find

$$n = 1{\cdot}9965.$$

This value of n is of importance, as it has been found without any assumption of the value of $a \times p$.

2.—Month of February.

Table V.—*Mean Hourly Observations.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	38°·8 F.	12	Noon.	43°·6 F.
1	1 A.M.	38 ·6 ,,	13	1 P.M.	44 ·3 ,,
2	2 ,,	38 ·4 ,,	14	2 ,,	44 ·3 ,,
3	3 ,,	38 ·3 ,,	15	3 ,,	44 ·0 ,,
4	4 ,,	38 ·1 ,,	16	4 ,,	43 ·2 ,,
5	5 ,,	38 ·2 ,,	17	5 ,,	42 ·0 ,,
6	6 ,,	38 ·1 ,,	18	6 ,,	41 ·0 ,,
7	7 ,,	38 ·3 ,,	19	7 ,,	40 ·1 ,,
8	8 ,,	38 ·8 ,,	20	8 ,,	39 ·8 ,,
9	9 ,,	39 ·7 ,,	21	9 ,,	39 ·4 ,,
10	10 ,,	40 ·9 ,,	22	10 ,,	39 ·1 ,,
11	11 ,,	42 ·5 ,,	23	11 ,,	38 ·9 ,,

This Table, plotted to scale, is represented in Plate 1, and the following Table is deduced from it, omitting individual figures, which are obviously erroneous:—

Table VI.—*Nocturnal Radiation* (February).

Time.	Temperature.	$\frac{d\theta}{dt}$.
5— 6 P.M.	41°·50 F.	− 1°·000
6— 8 ,,	40 ·30 ,,	− 0 ·600
8— 9 ,,	39 ·60 ,,	− 0 ·400
9—10 ,,	39 ·25 ,,	− 0 ·300
10—11 ,,	39 ·00 ,,	− 0 ·200
11— 2 A.M.	38 ·67 ,,	− 0 ·167
2— 3 ,,	38 ·40 ,,	− 0 ·100
3— 5 ,,	38 ·20 ,,	− 0 ·050

The day temperatures give the following:—

TABLE VII.—*Sunrise to Sunset* (February).

Hour.	θ.	$\frac{d\theta}{dt}$.	cos z.*
7 A.M.— 8 A.M.	38°·55 F.	+ 0·5	0·0652
8 ,, — 9 ,,	39 ·25 ,,	+ 0·9	0·1933
9 ,, —10 ,,	40 ·30 ,,	+ 1·2	0·3044
10 ,, —11 ,,	41 ·70 ,,	+ 1·6	0·3830
11 ,, —Noon.	43 ·05 ,,	+ 1·1	0·4237
Noon — 1 P.M.	43 ·95 ,,	+ 0·7	0·4237
1 ,, — 2 ,,	44 ·30 ,,	± 0·0	0·3830
2 ,, — 3 ,,	44 ·15 ,,	– 0·3	0·3044
3 ,, — 4 ,,	43 ·60 ,,	– 0·8	0·1933
4 ,, — 5 ,,	42 ·60 ,,	– 1·2	0·0652

Taking these observations, hour by hour, we have, as before,

$$S_1 + R_1 = +\ 0{\cdot}5,$$
$$S_2 + R_2 = +\ 0{\cdot}9,$$
$$S_3 + R_3 = +\ 1{\cdot}2,$$
$$S_4 + R_4 = +\ 1{\cdot}6,$$
$$S_5 + R_5 = +\ 1{\cdot}1,$$
$$S_5 + R_6 = +\ 0{\cdot}7,$$
$$S_4 + R_7 = \pm\ 0{\cdot}0,$$
$$S_3 + R_8 = -\ 0{\cdot}3,$$
$$S_2 + R_9 = -\ 0{\cdot}8,$$
$$S_1 + R_{10} = -\ 1{\cdot}2.$$

* $\cos z = -\ 0{\cdot}173 + 0{\cdot}607 \cos h.$
Sunrise = $7^h\ 7^m$ A.M.
Sunset = $4^h\ 53^m$ P.M.

Table VI. gives

$$R_{(38 \cdot 55)} = -0 \cdot 14,$$

$$R_{(39 \cdot 25)} = -0 \cdot 30,$$

$$R_{(40 \cdot 30)} = -0 \cdot 60,$$

$$R_{(41 \cdot 70)} = -1 \cdot 08.$$

Hence, we find

$$S_1 = 0 \cdot 64, \qquad a_1 = 9 \cdot 82,$$

$$S_2 = 1 \cdot 20, \qquad 6 \cdot 22,$$

$$S_3 = 1 \cdot 80, \qquad 5 \cdot 92,$$

$$S_4 = 2 \cdot 68, \qquad 7 \cdot 00.$$

We have, therefore,

$$a_1 = \mathbf{7 \cdot 24},$$

from which we calculate

$$S_5 = 3 \cdot 07,$$

$$R_{(53 \cdot 05)} = -1 \cdot 97,$$

$$R_{(63 \cdot 95)} = -2 \cdot 37.$$

We also have, from the four last equations,

$$R_{(44 \cdot 30)} = -2 \cdot 68,$$

$$R_{(44 \cdot 15)} = -2 \cdot 10,$$

$$R_{(43 \cdot 60)} = -2 \cdot 00,$$

$$R_{(42 \cdot 60)} = -1 \cdot 84.$$

Collecting together all the values of Radiation, in terms of temperature, we have—

TABLE VIII.—*Radiation* (February).

θ.	R.		θ.	R.	
38°·20 F.	− 0·05	Night.	42°·60 F.	− 1·84	Day.
38 ·40 ,,	− 0·10		43 ·05 ,,	− 1·97	
38 ·67 ,,	− 0·17		43 ·60 ,,	− 2·00	
39 ·00 ,,	− 0·20		43 ·95 ,,	− 2·37	
39 ·25 ,,	− 0·30		44 ·15 ,,	− 2·10	
39 ·60 ,,	− 0·40		44 ·30 ,,	− 2·68	
40 ·30 ,,	− 0·60				
41 ·50 ,,	− 1·00				

The lowest temperature, before sunrise, at which Radiation nearly ceased in February, was 38°·1 F.; from which we find

TABLE VIII. (*bis*).—*Radiation* (February).

	$(\theta - \theta_0)$.*	$\frac{d\theta}{dt}$.
Night,	0°·00 F.	− 0·05
	0 ·20 ,,	− 0·10
	0 ·47 ,,	− 0·17
	0 ·80 ,,	− 0·20
	1 ·05 ,,	− 0·30
	1 ·40 ,,	− 0·40
	2 ·10 ,,	− 0·60
	3 ·30 ,,	− 1·00
Day,	4 ·40 ,,	− 1·84
	4 ·85 ,,	− 1·97
	5 ·40 ,,	− 2·00
	5 ·75 ,,	− 2·37
	5 ·95 ,,	− 2·10
	6 ·10 ,,	− 2·68

* $\theta_0 = 38°·1$ F.

The night observations are represented by the right line

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}3,$$

or

$$d\theta = -\,0{\cdot}303\,(\theta - \theta_0).$$

The whole Table is plotted in Plate 6.

Combining the six day-radiations together, we have fifteen combinations, in pairs, to determine n and p. The mean values of all fifteen give

$$n = 1{\cdot}968, \qquad \log\,(p) = 1{\cdot}10738, \qquad p = 12{\cdot}805.$$

I have plotted in Plate 6 a portion ab of this parabolic curve for comparison with the observations.

The critical points of the curve of temperatures are shown in Plate 3, and may be discussed as follows:—

$$(a)\quad \frac{d\theta}{dt} = 0 \;\begin{cases} 5^{\text{h}}\ 0^{\text{m}}\ \text{A.M.}, & \theta = 38^{\circ}{\cdot}1\ \text{F.} \\ 1^{\text{h}}\ 54^{\text{m}}\ \text{P.M.}, & \theta = 44^{\circ}{\cdot}4\ \text{F.}, \qquad h = 28^{\circ}{\cdot}30'. \end{cases}$$

Hence we find

$$\theta_0 = 38^{\circ}{\cdot}1\ \text{F.},$$

and

$$ap\cos z = (\theta - \theta_0)^n,$$

or,

$$7{\cdot}24 \times 12{\cdot}805 \times 0{\cdot}361 = (6{\cdot}2)^n,$$

or,

$$n = 1{\cdot}924\,;$$

$$(b)\quad \frac{d^2\theta}{dt^2} = 0.$$

The two points of inflexion, Plate 1, or the maximum and minimum, Plate 3, give

$$\frac{d^2\theta}{dt^2} = 0 \quad \text{at} \quad \begin{cases} 10^{\text{h}}\ 30^{\text{m}}\ \text{A.M.}, & h = 22^\circ\,30', \\ \theta = 41^\circ{\cdot}7, & \theta - \theta_0 = \ 3{\cdot}5, \\ S + R = +\,1{\cdot}6, \end{cases}$$

and

$$\text{at} \quad \begin{cases} 4^{\text{h}}\ 30^{\text{m}}\ \text{P.M.}, & h = 67^\circ\,30', \\ \theta = 42^\circ{\cdot}60, & \theta - \theta_0 = \ 4{\cdot}40, \\ S + R = 1{\cdot}2. \end{cases}$$

Hence we have

$$\frac{ap\beta}{n} \times 0{\cdot}383 = (3{\cdot}5)^{n-1} \times 1{\cdot}6,$$

$$\frac{ap\beta}{n} \times 0{\cdot}922 = (4{\cdot}4)^{n-1} \times 1{\cdot}2;$$

or, dividing,

$$\frac{0{\cdot}383}{0{\cdot}922} = \left(\frac{3{\cdot}5}{4{\cdot}4}\right)^{n-1} \times \frac{1{\cdot}6}{1{\cdot}2},$$

which gives

$$n = 6{\cdot}096.$$

This value of n is quite erroneous, and the error arises from the fact that the temperatures at the points of inflexion, morning and evening, are nearly equal, so that their ratio approaches unity,* all powers of which are equal to unity; and therefore n becomes indeterminate, or its value critical.

* For $\left(\frac{35}{43}\right)^{n-1} = (0{\cdot}795)^{n-1}$; so that n is indeterminate.

3.—Month of March.

Table IX.—*Mean Hourly Observations of Temperature.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	39°·9 F.	12	Noon.	47°·8 F.
1	1 A.M.	39 ·8 ,,	13	1 P.M.	48 ·6 ,,
2	2 ,,	39 ·5 ,,	14	2 ,,	48 ·6 ,,
3	3 ,,	39 ·2 ,,	15	3 ,,	48 ·3 ,,
4	4 ,,	38 ·9 ,,	16	4 ,,	47 ·3 ,,
5	5 ,,	38 ·8 ,,	17	5 ,,	46 ·1 ,,
6	6 ,,	38 ·9 ,,	18	6 ,,	44 ·6 ,,
7	7 ,,	39 ·2 ,,	19	7 ,,	43 ·2 ,,
8	8 ,,	40 ·3 ,,	20	8 ,,	41 ·9 ,,
9	9 ,,	42 ·6 ,,	21	9 ,,	41 ·1 ,,
10	10 ,,	44 ·7 ,,	22	10 ,,	40 ·5 ,,
11	11 ,,	46 ·3 ,,	23	11 ,,	40 ·2 ,,

This Table is plotted to scale in Plate 1, and from it we deduce the following Table of Nocturnal Radiation :—

Table X.—*Nocturnal Radiation* (March).

Time.	Temperature.	$\frac{d\theta}{dt}$.
6 P.M.— 7 P.M.	43°·90 F.	- 1·4 F.
7 ,, — 8 ,,	42 ·55 ,,	- 1·3 ,,
8 ,, — 9 ,,	41 ·50 ,,	- 0·8 ,,
9 ,, —10 ,,	40 ·80 ,,	- 0·6 ,,
10 ,, —11 ,,	40 ·35 ,,	- 0·3 ,,
11 ,, —Midnight.	40 ·05 ,,	- 0·3 ,,
Midnight—5 A.M.	39 ·35 ,,	- 0·22 ,,

From the day temperatures we find—

Table XI.—*Sunrise to Sunset* (March).

Hour.	θ.	$\frac{d\theta}{dt}$.	cos z.*
6 A.M.— 7 A.M.	39°·05 F.	+ 0·3 F.	0·067
7 ,, — 8 ,,	39 ·75 ,,	+ 1·1 ,,	0·209
8 ,, — 9 ,,	41 ·45 ,,	+ 2·3 ,,	0·348
9 ,, —10 ,,	43 ·65 ,,	+ 2·1 ,,	0·462
10 ,, —11 ,,	45 ·50 ,,	+ 1·6 ,,	0·543
11 ,, —Noon.	47 ·05 ,,	+ 1·5 ,,	0·584
Noon — 1 P.M.	48 ·20 ,,	+ 0·8 ,,	0·584
1 ,, — 2 ,,	48 ·60 ,,	± 0·0 ,,	0·543
2 ,, — 3 ,,	48 ·45 ,,	− 0·3 ,,	0·462
3 ,, — 4 ,,	47 ·80 ,,	− 1·0 ,,	0·348
4 ,, — 5 ,,	46 ·70 ,,	− 1·2 ,,	0·209
5 ,, — 6 ,,	45 ·35 ,,	− 1·5 ,,	0·067

Hence we find, using a similar notation,

$$S_1 + R_{(39\cdot05)} = + 0\cdot3,$$
$$S_2 + R_{(39\cdot75)} = + 1\cdot1,$$
$$S_3 + R_{(41\cdot45)} = + 2\cdot3,$$
$$S_4 + R_{(43\cdot65)} = + 2\cdot1,$$
$$S_5 + R_{(45\cdot50)} = + 1\cdot6$$
$$S_6 + R_{(47\cdot05)} = + 1\cdot5,$$
$$S_6 + R_{(48\cdot20)} = + 0\cdot8,$$
$$S_5 + R_{(48\cdot60)} = \pm 0\cdot0,$$
$$S_4 + R_{(48\cdot45)} = - 0\cdot3,$$
$$S_3 + R_{(47\cdot80)} = - 1\cdot0,$$
$$S_2 + R_{(46\cdot70)} = - 1\cdot2,$$
$$S_1 + R_{(45\cdot35)} = - 1\cdot5.$$

* $\cos z = - 0\cdot0258 + 0\cdot622 \cos h.$
Sunrise = $6^h\ 16^m$ A.M.
Sunset = $6^h\ 6^m$ P.M.

R is known in the first four equations from Table X. (and therefore also S):

$$R_{(39\cdot05)} = -\,0\cdot11, \qquad S_1 = 0\cdot41,$$
$$R_{(39\cdot75)} = -\,0\cdot27, \qquad S_2 = 1\cdot37,$$
$$R_{(41\cdot45)} = -\,0\cdot78, \qquad S_3 = 3\cdot08,$$
$$R_{(43\cdot65)} = -\,1\cdot39, \qquad S_4 = 3\cdot49.$$

By the column of cosines we find,

$$\begin{array}{rl} a_1 = & 6\cdot12 \\ & 6\cdot55 \\ & 8\cdot85 \\ & 7\cdot55 \\ \hline & \mathbf{7\cdot27} \end{array}$$

Hence we find

$$S = 3\cdot95,$$
$$S_6 = 4\cdot25.$$

By these values we calculate

$$R_{(45\cdot50)} = -\,2\cdot35,$$
$$R_{(47\cdot05)} = -\,2\cdot75,$$
$$R_{(48\cdot20)} = -\,3\cdot45,$$
$$R_{(48\cdot60)} = -\,4\cdot17.$$

Also, from the last four equations,

$$R_{(45\cdot35)} = -\,1\cdot91,$$
$$R_{(46\cdot70)} = -\,2\cdot57,$$
$$R_{(47\cdot80)} = -\,4\cdot08,$$
$$R_{(48\cdot45)} = -\,3\cdot79.$$

The lowest temperature before sunrise was 38°·8, which I assume to be the value of θ_0. Collecting all the Radiations together, I find,

TABLE XII.—*Radiation* (March).

	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
Night,	0·55	− 0·22
	1·25	− 0·30
	1·55	− 0·30
	2·00	− 0·60
	2·70	− 0·80
	3·75	− 1·30
	5·10	− 1·40
Day,	6·55	− 1·91
	6·70	− 2·35
	7·90	− 2·57
	8·25	− 2·75
	9·00	− 4·08
	9·40	− 3·45
	9·65	− 3·79
	9·80	− 4·17

I resolve the day Radiations into three groups, as shown in Plate 6, viz.:

	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	6·02	− 2·13
(3·4)	8·07	− 2·66
(5·6·7·8)	9·46	− 3·87

From these I form three linear equations, which, combined in pairs, give the mean values of n and p.

$$n = 1{\cdot}72, \qquad \log(p) = 1{\cdot}103267, \qquad p = 12{\cdot}68.$$

With these values I have drawn on Plate 6 the parabolic curve that best represents the observations.

* $\theta_0 = 38^\circ{\cdot}8$ F.

The night Radiation is represented by the right line

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}643, \text{ or } d\theta = -\, 0{\cdot}274\,(\theta - \theta_0).$$

From $\dfrac{d\theta}{dt} = 0$, we have

$$\theta_0 = 38^\circ{\cdot}8 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n;$$

or

$$7{\cdot}27 \times 12{\cdot}68 \times 0{\cdot}543 = (9{\cdot}8)^n,$$

$$n = 1{\cdot}715.$$

From

$$\frac{d^2\theta}{dt^2} = 0,$$

$$\frac{ap\beta}{n} \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

we have

$$\frac{ap\beta}{n} \times 0{\cdot}793 = (2{\cdot}65)^{n-1} \times 2{\cdot}3,$$

$$\frac{ap\beta}{n} \times 0{\cdot}991 = (6{\cdot}55)^{n-1} \times 1{\cdot}5.$$

Dividing, we have

$$n = 1{\cdot}718.$$

Plate 3, shows the curve of hourly changes of temperature with its singular points.

4.—Month of April.

Table XIII.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	43°·9 F.	12	Noon.	55°·5 F.
1	1 A.M.	43 ·5 ,,	13	1 P.M.	56 ·6 ,,
2	2 ,,	43 ·0 ,,	14	2 ,,	56 ·9 ,,
3	3 ,,	42 ·5 ,,	15	3 ,,	56 ·4 ,,
4	4 ,,	42 ·1 ,,	16	4 ,,	55 ·4 ,,
5	5 ,,	42 ·0 ,,	17	5 ,,	54 ·1 ,,
6	6 ,,	42 ·7 ,,	18	6 ,,	52 ·2 ,,
7	7 ,,	44 ·4 ,,	19	7 ,,	49 ·8 ,,
8	8 ,,	46 ·7 ,,	20	8 ,,	48 ·0 ,,
9	9 ,,	49 ·6 ,,	21	9 ,,	46 ·7 ,,
10	10 ,,	51 ·9 ,,	22	10 ,,	45 ·5 ,,
11	11 ,,	54 ·0 ,,	23	11 ,,	44 ·6 ,,
Sunset, 7^h 0^m P.M.			Sunrise, 5^h 2^m A.M.		

This Table is plotted in Plate 1.

Table XIV.—*Night Radiation.*

Hour.	θ.	$\frac{d\theta}{dt}$.
7 P.M.— 8 P.M.	48°·90 F.	− 1·8
8 ,, — 9 ,,	47 ·35 ,,	− 1·3
9 ,, —10 ,,	46 ·10 ,,	− 1·2
10 ,, —11 ,,	45 ·05 ,,	− 0·9
11 ,, —Midnight	44 ·25 ,,	− 0·7
Midnight— 1 A.M.	43 ·70 ,,	− 0·4
1 A.M.— 2 ,,	43 ·25 ,,	− 0·5
2 ,, — 3 ,,	42 ·75 ,,	− 0·5
3 ,, — 4 ,,	42 ·30 ,,	− 0·4
4 ,, — 5 ,,	42 ·05 ,,	− 0·1

TABLE XV.—*Day Temperatures.*

Hour.	θ.	$\frac{d\theta}{dt}$.	cos z.*
5 A.M.— 6 A.M.	42°·35 F.	+ 0·7	0·068
6 ,, — 7 ,,	43 ·55 ,,	+ 1·7	0·215
7 ,, — 8 ,,	45 ·55 ,,	+ 2·3	0·369
8 ,, — 9 ,,	48 ·15 ,,	+ 2·9	0·506
9 ,, —10 ,,	50 ·75 ,,	+ 2·3	0·618
10 ,, —11 ,,	52 ·95 ,,	+ 2·1	0·697
11 ,, —Noon.	54 ·80 ,,	+ 1·5	0·739
Noon — 1 P.M.	56 ·05 ,,	+ 1·1	0·739
1 P.M.— 2 ,,	56 ·75 ,,	+ 0·3	0·697
2 ,, — 3 ,,	56 ·65 ,,	− 0·5	0·618
3 ,, — 4 ,,	55 ·90 ,,	− 1·0	0·506
4 ,, — 5 ,,	54 ·75 ,,	− 1·3	0·369
5 ,, — 6 ,,	53 ·15 ,,	− 1·9	0·215
6 ,, — 7 ,,	51 ·10 ,,	− 2·4	0·068

Hence we find fourteen equations, viz.,

$$S_1 + R_{(42\cdot35)} = +\ 0\cdot7,$$
$$S_2 + R_{(43\cdot55)} = +\ 1\cdot7,$$
$$S_3 + R_{(45\cdot55)} = +\ 2\cdot3,$$
$$S_4 + R_{(48\cdot15)} = +\ 2\cdot9,$$
$$S_5 + R_{(50\cdot75)} = +\ 2\cdot3,$$
$$S_6 + R_{(52\cdot95)} = +\ 2\cdot1,$$
$$S_7 + R_{(54\cdot80)} = +\ 1\cdot5,$$
$$S_7 + R_{(56\cdot05)} = +\ 1\cdot1,$$
$$S_6 + R_{(56\cdot75)} = +\ 0\cdot3,$$
$$S_5 + R_{(56\cdot65)} = -\ 0\cdot5,$$
$$S_4 + R_{(55\cdot90)} = -\ 1\cdot0,$$
$$S_3 + R_{(54\cdot75)} = -\ 1\cdot3,$$
$$S_2 + R_{(53\cdot15)} = -\ 1\cdot9,$$
$$S_1 + R_{(51\cdot00)} = -\ 2\cdot4.$$

* $\cos z = +\ 0\cdot136 + 0\cdot614 \cos h.$

From the first four equations we find, with the aid of Table XIV.,

$$R_{(42\cdot35)} = -\ 0\cdot10, \qquad S_1 = 0\cdot80,$$

$$R_{(43\cdot55)} = -\ 0\cdot42, \qquad S_2 = 2\cdot12,$$

$$R_{(45\cdot55)} = -\ 0\cdot94, \qquad S_3 = 3\cdot24,$$

$$R_{(48\cdot15)} = -\ 1\cdot62, \qquad S_4 = 4\cdot52.$$

Hence we find, from the last four equations,

$$R_{(51\cdot00)} = -\ 3\cdot20,$$

$$R_{(53\cdot15)} = -\ 4\cdot02,$$

$$R_{(54\cdot75)} = -\ 4\cdot54,$$

$$R_{(55\cdot90)} = -\ 5\cdot52.$$

With the aid of the column of cosines, we find,

$$\begin{array}{rl} a_1 = & 12\cdot18 \\ & 9\cdot86 \\ & 8\cdot78 \\ & 8\cdot94 \\ \hline & \mathbf{9\cdot94} \end{array}$$

Hence we find

$$S_5 = 6\cdot14, \quad R_{(50\cdot75)} = -\ 3\cdot84, \quad R_{(56\cdot05)} = -\ 6\cdot24,$$

$$S_6 = 6\cdot93, \quad R_{(52\cdot95)} = -\ 4\cdot83, \quad R_{(56\cdot75)} = -\ 6\cdot63,$$

$$S_7 = 7\cdot34, \quad R_{(54\cdot80)} = -\ 5\cdot84, \quad R_{(56\cdot65)} = -\ 6\cdot64.$$

Collecting together all the observations on Radiation, we find the following :—

Table XVI.—*Radiation* (April).

	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
Night,	0·05	− 0·10
	0·30	− 0·40
	0·75	− 0·50
	1·25	− 0·50
	1·70	− 0·40
	2·25	− 0·70
	3·05	− 0·90
	4·10	− 1·20
	5·35	− 1·30
	6·90	− 1·80
Day,	8·75	− 3·84
	9·00	− 3·20
	10·95	− 4·83
	11·15	− 4·02
	12·75	− 4·54
	12·80	− 5·84
	13·90	− 5·52
	14·05	− 6·24
	14·65	− 6·64
	14·85	− 6·63

From this Table it appears that the night Radiation is best represented by the right line passing through the origin

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}833\,;$$

or

$$d\theta = - \ 0{\cdot}261\,(\theta - \theta_0).$$

* Lowest temperature before Sunrise—
$\theta_0 = 42^\circ$ F.

The day Radiations may be divided into five natural groups, shown on Plate 6, viz.:

	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	8·87	− 3·52
(3·4)	11·05	− 4·42
(5·6)	12·77	− 5·19
(7·8)	13·97	− 5·88
(9·10)	14·75	− 6·63

These give five linear equations, with two unknown quantities, having ten combinations in pairs. The mean of all the values of n and p are—

$$n = 1{\cdot}352, \qquad \log(p) = 0{\cdot}763544, \qquad p = 5{\cdot}801.$$

Plate 3 gives the hourly changes of temperature.

From the critical points of the curve, we find—

$$(1) \quad \frac{d\theta}{dt} = 0, \quad \text{at} \quad 2^{\text{h}} \text{ P.M. } \theta_0 = 42^\circ{\cdot}0 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n,$$

$$10{\cdot}04 \times 5{\cdot}602 \times 0{\cdot}662 = (14{\cdot}9)^n,$$

$$n = 1{\cdot}342.$$

$$(2) \quad \frac{d^2\theta}{dt^2} = 0 \quad < \begin{matrix} 8^{\text{h}}\,30^{\text{m}} \text{ A.M.} \\ 6^{\text{h}}\,30^{\text{m}} \text{ P.M.} \end{matrix}$$

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0{\cdot}793 = (6{\cdot}15)^{n-1} \times 2{\cdot}9,$$

$$\frac{a\beta p}{n} \times 0{\cdot}991 = (9{\cdot}00)^{n-1} \times 2{\cdot}4,$$

$$n = 2{\cdot}020.$$

5.—Month of May.

Table XVII.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	49°·0 F.	12	Noon.	61°·1 F.
1	1 A.M.	48·4 ,,	13	1 P.M.	61 ·9 ,,
2	2 ,,	48·0 ,,	14	2 ,,	62 ·1 ,,
3	3 ,,	47·7 ,,	15	3 ,,	61 ·7 ,,
4	4 ,,	47·7 ,,	16	4 ,,	60 ·5 ,,
5	5 ,,	48·1 ,,	17	5 ,,	59 ·2 ,,
6	6 ,,	49·6 ,,	18	6 ,,	57 ·4 ,,
7	7 ,,	51·8 ,,	19	7 ,,	55 ·4 ,,
8	8 ,,	53·9 ,,	20	8 ,,	53 ·5 ,,
9	9 ,,	56·4 ,,	21	9 ,,	52 ·1 ,,
10	10 ,,	58·4 ,,	22	10 ,,	50 ·3 ,,
11	11 ,,	59·9 ,,	23	11 ,,	49 ·9 ,,
		Sunrise = $4^h\ 1^m$ A.M.		Sunset = $7^h\ 52^m$ P.M.	

This Table is plotted in Plate 1. From this we find the following :—

Table XVIII.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
3 A.M.— 4 A.M.	47°·70 F.	0·00	– 0·0
2 ,, — 3 ,,	47 ·85 ,,	0·15	– 0·3
1 ,, — 2 ,,	48 ·20 ,,	0·50	– 0·4
Midnight— 1 A.M.	48 ·70 ,,	1·00	– 0·6
11 P.M.—Midnight	49 ·45 ,,	1·75	– 0·9
10 ,, —11 P.M.	50 ·40 ,,	2·70	– 1·0
9 ,, —10 ,,	51 ·50 ,,	3·80	– 1·2
8 ,, — 9 ,,	52 ·80 ,,	5·10	– 1·4

* $\theta_0 = 47°·7$ F.

The day temperatures give us—

TABLE XIX.—*Day Temperatures.*

Hour.	θ.	$\frac{d\theta}{dt}$.	cos z.*
4 A.M.— 5 A.M.	47°·90 F.	+ 0·4	0·056
5 ,, — 6 ,,	48 ·85 ,,	+ 1·5	0·179
6 ,, — 7 ,,	50 ·70 ,,	+ 2·2	0·331
7 ,, — 8 ,,	52 ·85 ,,	+ 2·1	0·478
8 ,, — 9 ,,	55 ·15 ,,	+ 2·5	0·610
9 ,, —10 ,,	57 ·40 ,,	+ 2·0	0·718
10 ,, —11 ,,	59 ·15 ,,	+ 1·5	0·794
11 ,, —Noon.	60 ·50 ,,	+ 1·2	0·834
Noon — 1 P.M.	61 ·50 ,,	+ 0·8	0·834
1 P.M.— 2 ,,	62 ·00 ,,	+ 0·2	0·794
2 ,, — 3 ,,	61 ·90 ,,	− 0·4	0·718
3 ,, — 4 ,,	61 ·10 ,,	− 1·2	0·610
4 ,, — 5 ,,	59 ·85 ,,	− 1·3	0·478
5 ,, — 6 ,,	58 ·30 ,,	− 1·8	0·331
6 ,, — 7 ,,	56 ·40 ,,	− 2·0	0·179
7 ,, — 8 ,,	54 ·45 ,,	− 1·9	0·056

Hence we have sixteen equations, viz.:

$$S_1 + R_{(47\cdot90)} = +\ 0\cdot4,$$
$$S_2 + R_{(48\cdot85)} = +\ 1\cdot5,$$
$$S_3 + R_{(50\cdot70)} = +\ 2\cdot2,$$
$$S_4 + R_{(52\cdot85)} = +\ 2\cdot1,$$
$$S_5 + R_{(55\cdot15)} = +\ 2\cdot5,$$
$$S_6 + R_{(57\cdot40)} = +\ 2\cdot0,$$

* $\cos z = 0\cdot255 + 0\cdot589 \cos h$.

$$S_7 + R_{(59\cdot15)} = + 1\cdot5,$$

$$S_8 + R_{(60\cdot50)} = + 1\cdot2,$$

$$S_8 + R_{(61\cdot50)} = + 0\cdot8,$$

$$S_7 + R_{(62\cdot00)} = + 0\cdot2,$$

$$S_6 + R_{(61\cdot90)} = - 0\cdot4,$$

$$S_5 + R_{(61\cdot10)} = - 1\cdot2,$$

$$S_4 + R_{(59\cdot85)} = - 1\cdot3,$$

$$S_3 + R_{(58\cdot30)} = - 1\cdot8,$$

$$S_2 + R_{(56\cdot40)} = - 2\cdot0,$$

$$S_1 + R_{(54\cdot45)} = - 1\cdot9.$$

Solving the first four and last four equations, with the help of Table XVIII., we find

$$R_{(47\cdot90)} = - 0\cdot31, \quad S_1 = 0\cdot71, \quad R_{(54\cdot45)} = - 2\cdot61,$$

$$R_{(48\cdot85)} = - 0\cdot67, \quad S_2 = 2\cdot17, \quad R_{(56\cdot40)} = - 4\cdot17,$$

$$R_{(50\cdot70)} = - 1\cdot06, \quad S_3 = 3\cdot26, \quad R_{(58\cdot30)} = - 5\cdot06,$$

$$R_{(52\cdot85)} = - 1\cdot41, \quad S_4 = 3\cdot51, \quad R_{(59\cdot85)} = - 4\cdot81.$$

With the aid of the cosine column, we find

$$\begin{array}{rr} a_1 = & 12\cdot68 \\ & 12\cdot12 \\ & 9\cdot85 \\ & 7\cdot35 \\ \hline & 10\cdot50 \end{array}$$

The eight remaining equations give, with the aid of the cosine column:

$$S_5 = 5{\cdot}355,$$
$$S_6 = 7{\cdot}539,$$
$$S_7 = 8{\cdot}337,$$
$$S_8 = 8{\cdot}757.$$

$$R_{(55\cdot15)} = -\,2{\cdot}85, \quad R_{(57\cdot40)} = -\,5{\cdot}54,$$
$$R_{(59\cdot15)} = -\,6{\cdot}84, \quad R_{(60\cdot50)} = -\,7{\cdot}56,$$
$$R_{(61\cdot50)} = -\,7{\cdot}96, \quad R_{(62\cdot00)} = -\,8{\cdot}14,$$
$$R_{(61\cdot90)} = -\,7{\cdot}94, \quad R_{(61\cdot10)} = -\,6{\cdot}55.$$

Collecting all the values of day Radiation, we form the following :—

TABLE XX.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
54°·45 F.	6·75	− 2·61
55 ·15 ,,	7·45	− 2·85
56 ·40 ,,	8·70	− 4·17
57 ·40 ,,	9·70	− 5·54
58 ·30 ,,	10·60	− 5·06
59 ·15 ,,	11·45	− 6·84
59 ·85 ,,	12·15	− 4·81
60 ·50 ,,	12·80	− 7·56
61 ·10 ,,	13·40	− 6.55
61 ·50 ,,	13·80	− 7·96
61 ·90 ,,	14·20	− 7·94
62 ·00 ,,	14·30	− 8·14

These observations are plotted in Plate 7, and divide themselves into the following natural groups, viz. :—

	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	7·10	− 2·73
(3·4·5)	9·67	− 4·92
(6·7)	11·80	− 5·82
(8·9)	13·10	− 7·05
(10·11·12)	14·10	− 8·01

These give five equations and ten combinations to determine n and p; the mean of all are—

$$n = 1{\cdot}520, \qquad \log(p) = 0{\cdot}843916, \qquad p = 6{\cdot}981.$$

I have plotted the parabolic curve *ab* in Plate 7 for comparison with the observations. The night Radiation is represented by the right line,

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}643\,;$$

or

$$d\theta = -\,0{\cdot}274\,(\theta - \theta_0).$$

Plate 4, shows the curve of changes of hourly temperature, from which we find—

$$\frac{d\theta}{dt} = 0 < \begin{array}{ll} 3^h\,30^m \text{ A.M.}, & \theta_0 = 47^\circ{\cdot}7, \\ 2^h\,0^m \text{ P.M.}, & \theta\ = 62^\circ{\cdot}1. \end{array}$$

Hence $\qquad \theta_0 = 47^\circ{\cdot}7, \quad$ and $\quad ap \cos z = (14{\cdot}4'');$

or

$$10{\cdot}50 \times 6{\cdot}981 \times 0{\cdot}866 = (14{\cdot}4''),$$

or

$$n = 1{\cdot}556.$$

$$\frac{d^2\theta}{dt^2} = 0 < \begin{array}{ll} 8^h\,30^m \text{ A.M.}, & h = 52^\circ{\cdot}30', \\ 6^h\,30^m \text{ P.M.}, & h = 107^\circ{\cdot}30'. \end{array}$$

$$\frac{\alpha\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{\alpha\beta p}{n} \times 0{\cdot}793 = (7{\cdot}45)^{n-1} \times 2{\cdot}5,$$

$$\frac{\alpha\beta p}{n} \times 0{\cdot}991 = (8{\cdot}7)^{n-1} \times 2{\cdot}0;$$

or, dividing,

$$0{\cdot}800 = (0{\cdot}856)^{n-1} \times 1{\cdot}25.$$

This gives $$n = 3{\cdot}869.$$

This value of n (like the similar value for February) is critical, because the ratio

$$\left(\frac{7{\cdot}45}{8{\cdot}70}\right)^{n-1} = (0{\cdot}856)^{n-1}$$

so nearly approaches unity, that the value of n becomes indeterminate.

An inspection of Plate 4, however, shows that while the afternoon point of inflexion is accurately fixed by the Table XIX., the forenoon point of inflexion requires a correction which the Plate supplies, viz.:

Forenoon point of inflexion—

$$\text{at } 7^{h}\ 48^{m} \text{ A.M., } \theta = 53^{\circ}{\cdot}48 \text{ F.,} \qquad S + R = 2{\cdot}58,$$

instead of

$$\text{at } 8^{h}\ 30^{m} \text{ A.M., } \theta = 55^{\circ}{\cdot}15 \text{ ,,} \qquad S + R = 2{\cdot}50.$$

Using these values for the forenoon point of inflexion, and the old values for the afternoon point, we find

$$n = 1{\cdot}884.$$

6.—Month of June.

Table XXI.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	54°·4 F.	12	Noon.	67°·9 F.
1	1 A.M.	53 ·5 ,,	13	1 P.M.	68 ·7 ,,
2	2 ,,	52 ·6 ,,	14	2 ,,	69 ·2 ,,
3	3 ,,	51 ·9 ,,	15	3 ,,	69 ·0 ,,
4	4 ,,	51 ·3 ,,	16	4 ,,	68 ·0 ,,
5	5 ,,	51 ·8 ,,	17	5 ,,	66 ·7 ,,
6	6 ,,	54 ·2 ,,	18	6 ,,	65 ·1 ,,
7	7 ,,	57 ·6 ,,	19	7 ,,	63 ·0 ,,
8	8 ,,	60 ·6 ,,	20	8 ,,	60 ·6 ,,
9	9 ,,	63 ·1 ,,	21	9 ,,	58 ·8 ,,
10	10 ,,	65 ·1 ,,	22	10 ,,	57 ·0 ,,
11	11 ,,	66 ·4 ,,	23	11 ,,	55 ·6 ,,
Sunrise = $3^h\ 33^m$ A.M.			Sunset = $8^h\ 27^m$ P.M.		

This Table is plotted in Plate 1.

Table XXII.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
2 A.M.— 3 A.M.	52°·25 F.	0·95	− 0·7
1 ,, — 2 ,,	53 ·05 ,,	1·75	− 0·9
Midnight— 1 ,,	53 ·95 ,,	2·65	− 0·9
11 P.M.—Midnight	55 ·00 ,,	3·70	− 1·2
10 ,, —11 P.M.	56 ·30 ,,	5·00	− 1·4
9 ,, —10 ,,	57 ·90 ,,	6·60	− 1·8

* $\theta_0 = 51°·3$ F.

From this Table I find that the night Radiation is represented by the right line

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}666;$$

or

$$d\theta = - 0{\cdot}273 \, (\theta - \theta_0).$$

The day temperatures give us—

Table XXIII.—*Day Temperatures.*

Hour.	θ.	$\frac{d\theta}{dt}$.	cos z.*
4 A.M.— 5 A.M.	51°·55 F.	+ 0·5	0·097
5 ,, — 6 ,,	53 ·00 ,,	+ 2·4	0·235
6 ,, — 7 ,,	55 ·90 ,,	+ 3·4	0·383
7 ,, — 8 ,,	59 ·10 ,,	+ 3·0	0·526
8 ,, — 9 ,,	61 ·85 ,,	+ 2·5	0·654
9 ,, —10 ,,	64 ·10 ,,	+ 2·0	0·758
10 ,, —11 ,,	65 ·75 ,,	+ 1·3	0·832
11 ,, —Noon	67 ·15 ,,	+ 1·5	0·871
Noon — 1 P.M.	68 ·30 ,,	+ 0·8	0·871
1 P.M.— 2 ,,	68 ·95 ,,	+ 0·5	0·832
2 ,, — 3 ,,	69 ·10 ,,	- 0·2	0·758
3 ,, — 4 ,,	68 ·50 ,,	- 1·0	0·654
4 ,, — 5 ,,	67 ·35 ,,	- 1·3	0·526
5 ,, — 6 ,,	65 ·90 ,,	- 1·6	0·383
6 ,, — 7 ,,	64 ·05 ,,	- 2·1	0·235
7 ,, — 8 ,,	61 ·80 ,,	- 2·4	0·097

* $\cos z = 0{\cdot}309 + 0{\cdot}572 \cos h$.

Hence we have sixteen equations, viz.:

$$
\begin{aligned}
S_1 + R_{(51\cdot55)} &= +\ 0\cdot5,\\
S_2 + R_{(53\cdot00)} &= +\ 2\cdot4,\\
S_3 + R_{(55\cdot90)} &= +\ 3\cdot4,\\
S_4 + R_{(59\cdot10)} &= +\ 3\cdot0,\\
S_5 + R_{(61\cdot85)} &= +\ 2\cdot5,\\
S_6 + R_{(64\cdot10)} &= +\ 2\cdot0,\\
S_7 + R_{(65\cdot75)} &= +\ 1\cdot3,\\
S_8 + R_{(67\cdot15)} &= +\ 1\cdot5,\\
S_8 + R_{(68\cdot30)} &= +\ 0\cdot8,\\
S_7 + R_{(68\cdot95)} &= +\ 0\cdot5,\\
S_6 + R_{(69\cdot10)} &= -\ 0\cdot2,\\
S_5 + R_{(68\cdot50)} &= -\ 1\cdot0,\\
S_4 + R_{(67\cdot35)} &= -\ 1\cdot3,\\
S_3 + R_{(65\cdot90)} &= -\ 1\cdot6,\\
S_2 + R_{(64\cdot05)} &= -\ 2\cdot1,\\
S_1 + R_{(61\cdot80)} &= -\ 2\cdot4.
\end{aligned}
$$

From Table XXII. we have

$$
\begin{aligned}
R_{(51\cdot55)} &= -\ 0\cdot16,\\
R_{(53\cdot00)} &= -\ 0\cdot89,\\
R_{(55\cdot90)} &= -\ 1\cdot33,\\
R_{(59\cdot10)} &= -\ 2\cdot08.
\end{aligned}
$$

From the first and last four equations we have

$$
\begin{aligned}
S_1 &= 0\cdot66, & R_{(61\cdot80)} &= -\ 3\cdot06,\\
S_2 &= 3\cdot29, & R_{(64\cdot05)} &= -\ 5\cdot39,\\
S_3 &= 4\cdot73, & R_{(65\cdot90)} &= -\ 6\cdot33,\\
S_4 &= 5\cdot08, & R_{(67\cdot35)} &= -\ 6\cdot38.
\end{aligned}
$$

From the column of cosines we have

$$
\begin{array}{rr}
a_1 = & 6\cdot81\\
 & 14\cdot00\\
 & 18\cdot12\\
 & 9\cdot66\\
\hline
 & \mathbf{12\cdot15}
\end{array}
$$

With the aid of the cosine column we find

$$S_5 = 7{\cdot}94,$$
$$S_6 = 9{\cdot}21,$$
$$S_7 = 10{\cdot}11,$$
$$S_8 = 10{\cdot}58;$$

and, finally, from the eight remaining equations,

$$R_{(61\cdot85)} = -5{\cdot}44, \qquad R_{(68\cdot30)} = -9{\cdot}78,$$
$$R_{(64\cdot10)} = -7{\cdot}21, \qquad R_{(68\cdot95)} = -9{\cdot}61,$$
$$R_{(65\cdot75)} = -8{\cdot}81, \qquad R_{(69\cdot10)} = -9{\cdot}41,$$
$$R_{(67\cdot15)} = -9{\cdot}08, \qquad R_{(68\cdot50)} = -8{\cdot}94.$$

Collecting all the values together we find the following :—

TABLE XXIV.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
61°·80 F.	10·50	- 3·06
61 ·85 ,,	10·55	- 5·44
64 ·05 ,,	12·75	- 5·39
64 ·10 ,,	12·80	- 7·21
65 ·75 ,,	14·45	- 8·81
65 ·90 ,,	14·60	- 6·33
67 ·15 ,,	15·85	- 9·08
67 ·35 ,,	16·05	- 6·78
68 ·30 ,,	17·00	- 9·78
68 ·50 ,,	17·20	- 8·94
68 ·95 ,,	17·65	- 9·61
69 ·10 ,,	17·80	- 9·41

These figures are plotted on Plate 7, which shows that they are arranged in six pairs, each pair having nearly the same temperature. Taking the means of each pair, Table XXIV. reduces to six observations, viz. :

Day Radiation.

No.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	10·52	− 4·25
(3·4)	12·77	− 6·30
(5·6)	14·82	− 7·57
(7·8)	15·95	− 7·93
(9·10)	17·10	− 9·36
(11·12)	17·72	− 9·51

This Table gives six linear equations of the form

$$n \log(\theta - \theta_0) = \log(p) + \log \frac{d\theta}{dt},$$

derived from the parabolic curve

$$(\theta - \theta_0)^n = p \frac{d\theta}{dt}.$$

There are fifteen combinations of these six equations, and from each combination the values of n and p may be found. The mean of all gives

$$n = 1·433, \qquad \log p = 0·8063333, \qquad p = 6·40.$$

With these values I have drawn the curve *ab* in Plate 7. From the singular points of the curve we have

$$\frac{d\theta}{dt} = 0, \quad \theta_0 = 51°·3 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n \quad \text{at} \quad 2^{\text{h}}\,12^{\text{m}} \text{ P.M.},$$

or $$12·15 \times 6·40 \times 0·789 = (17·9)^n, \qquad \text{or} \quad n = 1·427,$$

$$\frac{d^2\theta}{dt^2} = 0 \quad \text{at} \quad \begin{cases} 6^{\text{h}}\,30^{\text{m}} \text{ A.M.} \\ 7^{\text{h}}\,30^{\text{m}} \text{ P.M.} \end{cases}$$

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0·991 = (4·6)^{n-1} \times 3·4,$$

$$\frac{a\beta p}{n} \times 0·924 = (10·5)^{n-1} \times 2·4,$$

or $$n = 1·337$$

7.—Month of July.

Table XXV.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	58°·9 F.	12	Noon.	70°·3 F.
1	1 A.M.	58 ·4 ,,	13	1 P.M.	70 ·6 ,,
2	2 ,,	57 ·9 ,,	14	2 ,,	70 ·6 ,,
3	3 ,,	57 ·5 ,,	15	3 ,,	70 ·4 ,,
4	4 ,,	57 ·3 ,,	16	4 ,,	69 ·7 ,,
5	5 ,,	57 ·7 ,,	17	5 ,,	68 ·8 ,,
6	6 ,,	59 ·4 ,,	18	6 ,,	67 ·4 ,,
7	7 ,,	61 ·4 ,,	19	7 ,,	65 ·4 ,,
8	8 ,,	63 ·9 ,,	20	8 ,,	63 ·6 ,,
9	9 ,,	65 ·9 ,,	21	9 ,,	62 ·0 ,,
10	10 ,,	67 ·9 ,,	22	10 ,,	60 ·6 ,,
11	11 ,,	69 ·3 ,,	23	11 ,,	59 ·7 ,,
		Sunrise = $3^h\ 33^m$ A.M.		Sunset = $8^h\ 27^m$ P.M.	

This Table is plotted in Plate 2, and from it I find

Table XXVI.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
2 A.M.— 3 A.M.	57°·70 F.	0·40	− 0·4
1 ,, — 2 ,,	58 ·15 ,,	0·85	− 0·5
Midnight— 1 ,,	58 ·65 ,,	1·35	− 0·5
11 P.M.—Midnight	59 ·30 ,,	2·00	− 0·8
10 ,, —11 P.M.	60 ·15 ,,	2·85	− 0·9
9 ,, —10 ,,	61 ·30 ,,	4·00	− 1·4

* $\theta_0 = 57°·3$ F.

From this Table I find that the night Radiation is represented by the right line

$$\frac{\theta - \theta_0}{d\theta} = 2{\cdot}857\,;$$

or

$$d\theta = -\,0{\cdot}350\,(\theta - \theta_0).$$

The day temperatures give us

TABLE XXVII.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	cos z.*
4 A.M.— 5 A.M.	57°·50 F.	0·20	+ 0·4	0·070
5 ,, — 6 ,,	58 ·55 ,,	1·25	+ 1·4	0·211
6 ,, — 7 ,,	60 ·40 ,,	3·10	+ 2·0	0·361
7 ,, — 8 ,,	62 ·65 ,,	5·35	+ 2·5	0·506
8 ,, — 9 ,,	64 ·90 ,,	7·60	+ 2·0	0·636
9 ,, —10 ,,	66 ·90 ,,	9·60	+ 2·0	0·742
10 ,, —11 ,,	68 ·60 ,,	11·30	+ 1·4	0·817
11 ,, —Noon	69 ·80 ,,	12·50	+ 1·0	0·856
Noon — 1 P.M.	70 ·45 ,,	13·15	+ 0·3	0·856
1 P.M.— 2 ,,	70 ·60 ,,	13·30	± 0·0	0·817
2 ,, — 3 ,,	70 ·50 ,,	13·20	− 0·2	0·742
3 ,, — 4 ,,	70 ·05 ,,	12·75	− 0·7	0·636
4 ,, — 5 ,,	69 ·25 ,,	11·95	− 0·9	0·506
5 ,, — 6 ,,	68 ·10 ,,	10·80	− 1·4	0·361
6 ,, — 7 ,,	66 ·40 ,,	9·10	− 2·0	0·211
7 ,, — 8 ,,	64 ·50 ,,	7·20	− 1·8	0·070

* $\cos z = 0{\cdot}286 + 0{\cdot}580 \cos h$.

Hence we have sixteen equations, viz.:

$$
\begin{aligned}
S_1 + R_{(57\cdot50)} &= +\ 0\cdot4,\\
S_2 + R_{(58\cdot55)} &= +\ 1\cdot4,\\
S_3 + R_{(60\cdot40)} &= +\ 2\cdot0,\\
S_4 + R_{(62\cdot65)} &= +\ 2\cdot5,\\
S_5 + R_{(64\cdot90)} &= +\ 2\cdot0,\\
S_6 + R_{(66\cdot90)} &= +\ 2\cdot0,\\
S_7 + R_{(68\cdot60)} &= +\ 1\cdot4,\\
S_8 + R_{(69\cdot80)} &= +\ 1\cdot0,\\
S_8 + R_{(70\cdot45)} &= +\ 0\cdot3,\\
S_7 + R_{(70\cdot60)} &= \pm\ 0\cdot0,\\
S_6 + R_{(70\cdot50)} &= -\ 0\cdot2,\\
S_5 + R_{(70\cdot05)} &= -\ 0\cdot7,\\
S_4 + R_{(69\cdot25)} &= -\ 0\cdot9,\\
S_3 + R_{(68\cdot10)} &= -\ 1\cdot4,\\
S_2 + R_{(66\cdot40)} &= -\ 2\cdot0,\\
S_1 + R_{(64\cdot50)} &= -\ 1\cdot6.
\end{aligned}
$$

From Table XXVI. we have

$$
\begin{aligned}
R_{(57\cdot50)} &= -\ 0\cdot07, & R_{(58\cdot55)} &= -\ 0\cdot43,\\
R_{(60\cdot40)} &= -\ 1\cdot07, & R_{(62\cdot65)} &= -\ 1\cdot87.
\end{aligned}
$$

Hence from the first and last four equations we have

$$
\begin{aligned}
S_1 &= 0\cdot47, & R_{(64\cdot50)} &= -\ 2\cdot07,\\
S_2 &= 1\cdot83, & R_{(66\cdot40)} &= -\ 3\cdot83,\\
S_3 &= 3\cdot07, & R_{(68\cdot10)} &= -\ 4\cdot47,\\
S_4 &= 4\cdot37, & R_{(69\cdot25)} &= -\ 5\cdot27.
\end{aligned}
$$

From the column of cosines we have

$$
\begin{array}{rll}
a_1 = & 6\cdot71 & (S_1),\\
& 8\cdot67 & (S_2),\\
& 8\cdot50 & (S_3),\\
& 8\cdot64 & (S_4).\\
\hline
& \mathbf{8\cdot13} &
\end{array}
$$

Hence we find

$$S_5 = 5{\cdot}17, \qquad S_7 = 6{\cdot}64,$$

$$S_6 = 6{\cdot}03, \qquad S_8 = 6{\cdot}96.$$

By means of these values we find from the remaining eight equations—

$$R_{(64\cdot 90)} = -\,3{\cdot}17, \quad R_{(70\cdot 05)} = -\,5{\cdot}87,$$

$$R_{(66\cdot 90)} = -\,4{\cdot}03, \quad R_{(70\cdot 50)} = -\,6{\cdot}23,$$

$$R_{(68\cdot 60)} = -\,5{\cdot}24, \quad R_{(70\cdot 60)} = -\,6{\cdot}64,$$

$$R_{(69\cdot 80)} = -\,5{\cdot}96, \quad R_{(70\cdot 45)} = -\,6{\cdot}66.$$

Collecting together all the foregoing we find

TABLE XXVIII.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
64°·50 F.	7·20	– 2·07
64 ·90 ,,	7·60	– 3·17
66 ·40 ,,	9·10	– 3·83
66 ·90 ,,	9·60	– 4·03
68 ·10 ,,	10·80	– 4·47
68 ·60 ,,	11·30	– 5·24
69 ·25 ,,	11·95	– 5·27
69 ·80 ,,	12·50	– 5·96
70 ·05 ,,	12·75	– 5·87
70 ·45 ,,	13·15	– 6·66
70 ·50 ,,	13·20	– 6·23
70 ·60 ,,	13·30	– 6·64

These observations are plotted in Plate 8, and may be grouped as follows, in five sections, viz.:—

No.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	7·40	− 2·62
(3·4)	9·35	− 3·93
(5·6·7)	11·35	− 4·933
(8·9)	12·625	− 5·915
(10·11·12)	13·217	− 6·51

These five equations give, as before, ten combinations, in pairs, to determine n and p. The mean values of all give

$$n = 1·594, \quad \log(p) = 0·973378, \quad p = 9·4054.$$

From these values I have constructed the curve $a\,b$, in Plate 8, for comparison with the Radiation observations.

From the singular points of the curve we have

$$\frac{d\theta}{dt} = 0, \quad \theta_0 = 57°·3 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n \text{ at } 1^h\ 30^m \text{ P.M.},$$

or

$$8·13 \times 9·4054 \times 0·817 = (13·3)^n,$$

or

$$n = 1·450.$$

$$\frac{d^2\theta}{dt^2} = 0 \text{ at } \begin{cases} 7^h\ 30^m \text{ A.M.} \\ 6^h\ 30^m \text{ P.M.} \end{cases}$$

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0·924 = (5·35)^{n-1} \times 2·5,$$

$$\frac{a\beta p}{n} \times 0·991 = (9·10)^{n-1} \times 2·0,$$

or, dividing,

$$0·9325 = (0·588)^{n-1} \times 1·25,$$

or

$$n = 1·552.$$

8.—Month of August.

Table XXIX.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	57°·5 F.	12	Noon.	69°·1 F.
1	1 A.M.	57 ·1 ,,	13	1 P.M.	70 ·1 ,,
2	2 ,,	56 ·6 ,,	14	2 ,,	70 ·3 ,,
3	3 ,,	56 ·3 ,,	15	3 ,,	69 ·6 ,,
4	4 ,,	56 ·1 ,,	16	4 ,,	68 ·1 ,,
5	5 ,,	56 ·1 ,,	17	5 ,,	66 ·2 ,,
6	6 ,,	57 ·1 ,,	18	6 ,,	64 ·6 ,,
7	7 ,,	59 ·3 ,,	19	7 ,,	63 ·1 ,,
8	8 ,,	61 ·7 ,,	20	8 ,,	61 ·6 ,,
9	9 ,,	64 ·2 ,,	21	9 ,,	60 ·2 ,,
10	10 ,,	66 ·1 ,,	22	10 ,,	59 ·3 ,,
11	11 ,,	68 ·0 ,,	23	11 ,,	58 ·3 ,,
		Sunrise, $4^h\ 42^m$ A.M.		Sunset, $7^h\ 25^m$ P.M.	

This Table is plotted in Plate 2, and we find from it

Table XXX.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
8 P.M.— 9 P.M.	60°·90 F.	4·80	− 1·4
9 ,, — 10 ,,	59 ·75 ,,	3·65	− 0·9
10 ,, — 11 ,,	58 ·80 ,,	2·70	− 1·0
11 ,, —Midnight	57 ·90 ,,	1·80	− 0·8
Midnight— 1 A.M.	57 ·30 ,,	1·20	− 0·4
1 ,, — 2 ,,	56 ·85 ,,	0·75	− 0·5
2 ,, — 3 ,,	56 ·45 ,,	0·35	− 0·3
3 ,, — 4 ,,	56 ·20 ,,	0·10	− 0·2

* $\theta_0 = 56°{\cdot}1$ F.

From this Table we find

$$\frac{\theta - \theta_0}{d\theta} = 3{\cdot}429,$$

$$d\theta = -\ 0{\cdot}292\,(\theta - \theta_0).$$

The day temperatures give

TABLE XXXI.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	cos z.*
5 A.M.— 6 A.M.	56°·60 F.	0·50	+ 1·0	0·109
6 ,, — 7 ,,	58 ·20 ,,	2·10	+ 2·2	0·266
7 ,, — 8 ,,	60 ·50 ,,	4·40	+ 2·4	0·417
8 ,, — 9 ,,	62 ·95 ,,	6·85	+ 2·5	0·555
9 ,, —10 ,,	65 ·15 ,,	9·05	+ 1·9	0·664
10 ,, —11 ,,	67 ·05 ,,	10·95	+ 1·9	0·742
11 ,, —Noon.	68 ·55 ,,	12·45	+ 1·1	0·787
Noon — 1 P.M.	69 ·60 ,,	13·50	+ 1·0	0·787
1 P.M.— 2 ,,	70 ·20 ,,	14·10	+ 0·2	0·742
2 ,, — 3 ,,	69 ·95 ,,	13·85	- 0·7	0·664
3 ,, — 4 ,,	68 ·85 ,,	12·75	- 1·5	0·555
4 ,, — 5 ,,	67 ·15 ,,	11·05	- 1·9	0·417
5 ,, — 6 ,,	65 ·40 ,,	9·30	- 1·6	0·266
6 ,, — 7 ,,	63 ·85 ,,	7·75	- 1·5	0·109

* $\cos z = 0{\cdot}188 + 0{\cdot}605 \cos h.$

Hence we have fourteen equations, viz.:

$$
\begin{aligned}
S_1 + R_{(56\cdot60)} &= +\,1\cdot0,\\
S_2 + R_{(58\cdot20)} &= +\,2\cdot2,\\
S_3 + R_{(60\cdot50)} &= +\,2\cdot4,\\
S_4 + R_{(62\cdot95)} &= +\,2\cdot5,\\
S_5 + R_{(65\cdot15)} &= +\,1\cdot9,\\
S_6 + R_{(67\cdot05)} &= +\,1\cdot9,\\
S_7 + R_{(68\cdot55)} &= +\,1\cdot1,\\
S_7 + R_{(69\cdot60)} &= +\,1\cdot0,\\
S_6 + R_{(70\cdot20)} &= +\,0\cdot2,\\
S_5 + R_{(69\cdot95)} &= -\,0\cdot7,\\
S_4 + R_{(68\cdot85)} &= -\,1\cdot5,\\
S_3 + R_{(67\cdot15)} &= -\,1\cdot9,\\
S_2 + R_{(65\cdot40)} &= -\,1\cdot6,\\
S_1 + R_{(63\cdot85)} &= -\,1\cdot5.
\end{aligned}
$$

From the night Radiation we have

$$R_{(56\cdot60)} = -\,0\cdot13, \qquad R_{(58\cdot20)} = -\,0\cdot61. \qquad R_{(60\cdot50)} = -\,1\cdot28.$$

Hence we find, from the first three,

$$
\begin{array}{ll}
S_1 = 1\cdot13, & a_1 = 10\cdot40\\
S_2 = 2\cdot81, & 10\cdot50\\
S_3 = 3\cdot68, & \underline{8\cdot80}\\
 & \mathbf{9\cdot90};
\end{array}
$$

and from the last three equations, by means of the cosine column, we find,

$$S_4 = 5\cdot49. \qquad S_5 = 6\cdot57, \qquad S_6 = 7\cdot35, \qquad S_7 = 7\cdot79.$$

Hence we find

$$
\begin{array}{ll}
R_{(62\cdot95)} = -\,3\cdot99, & R_{(65\cdot15)} = -\,4\cdot67,\\
R_{(67\cdot05)} = -\,5\cdot45, & R_{(68\cdot55)} = -\,6\cdot69,\\
R_{(69\cdot60)} = -\,6\cdot79, & R_{(70\cdot20)} = -\,7\cdot15,\\
R_{(69\cdot95)} = -\,7\cdot27, & R_{(68\cdot85)} = -\,6\cdot99.
\end{array}
$$

Collecting all these together, we have

TABLE XXXII.—*Day Radiation.*

θ	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
62°·95 F.	6·85	− 3·99
63 ·85 ,,	7·75	− 2·63
65 ·15 ,,	9·05	− 4·67
65 ·40 ,,	9·30	− 4·41
67 ·05 ,,	10·95	− 5·45
67 ·15 ,,	11·05	− 5·58
68 ·55 ,,	12·45	− 6·69
68 ·85 ,,	12·75	− 6·99
69 ·60 ,,	13·50	− 6·79
69 ·95 ,,	13·85	− 7·27
70 ·20 ,,	14·10	− 7·15

These observations are plotted in Plate 8, and may be discussed in the following five groups :—

No.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	7·300	− 3·310
(3·4)	9·175	− 4·540
(5·6)	11·000	− 5·515
(7·8)	12·600	− 6·840
(9·10·11)	13·817	− 7·070

From these equations I have ten combinations to find the most probable values of n and p, which (including all the observations) are,

$$n = 1{\cdot}488, \qquad \log(p) = 0{\cdot}80104, \qquad p = 6{\cdot}325.$$

With these values I have constructed the curve *ab* in Plate 8, for comparison with the Radiation observations.

From the singular points of the curve we have

$$\frac{d\theta}{dt} = 0, \;\; \theta_0 = 56^\circ{\cdot}1 \text{ F.},$$

and

$$ap \cos z = (\theta - \theta_0)^n \quad \text{at} \quad 2^{\text{h}} \text{ P.M.}$$

This gives us the equation

$$9{\cdot}90 \times 6{\cdot}325 \times 0{\cdot}742 = (14{\cdot}2)^n,$$

or

$$n = 1{\cdot}447.$$

$$\frac{d^2\theta}{dt^2} = 0 < \begin{matrix} 8^{\text{h}}\ 30^{\text{m}} \text{ A.M.}, \\ 4^{\text{h}}\ 30^{\text{m}} \text{ P.M.}, \end{matrix}$$

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0{\cdot}793 = (6{\cdot}85)^{n-1} \times 2{\cdot}50,$$

$$\frac{a\beta p}{n} \times 0{\cdot}924 = (11{\cdot}05)^{n-1} \times 1{\cdot}90.$$

Taking the logarithms of both sides, and substracting, we find

$$n - 1 = 0{\cdot}898,$$

$$n = 1{\cdot}898.$$

The singular points are well shown in the derived curve, Plate 4.

9.—MONTH OF SEPTEMBER.

TABLE XXXIII.—*Mean Hourly Temperatures.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	53°·9 F.	12	Noon.	64°·3 F.
1	1 A.M.	53·4 ,,	13	1 P.M.	65 ·0 ,,
2	2 ,,	52·4 ,,	14	2 ,,	65 ·0 ,,
3	3 ,,	51·5 ,,	15	3 ,,	64 ·5 ,,
4	4 ,,	51·3 ,,	16	4 ,,	63 ·4 ,,
5	5 ,,	51·7 ,,	17	5 ,,	62 ·1 ,,
6	6 ,,	52·6 ,,	18	6 ,,	60 ·4 ,,
7	7 ,,	53·9 ,,	19	7 ,,	58 ·5 ,,
8	8 ,,	55·8 ,,	20	8 ,,	56 ·9 ,,
9	9 ,,	58·3 ,,	21	9 ,,	56 ·1 ,,
10	10 ,,	60·9 ,,	22	10 ,,	55 ·2 ,,
11	11 ,,	62·9 ,,	23	11 ,,	54 ·5 ,,
Sunrise = 5^h 36^m A.M.			Sunset = 6^h 12^m P.M.		

This Table is plotted in Plate 2, and from it we find

TABLE XXXIV.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$*.	$\frac{d\theta}{dt}$.
6 P.M.— 7 P.M.	59°·45 F.	8°·15 F.	− 1·9
7 ,, — 8 ,,	57 ·70 ,,	6 ·40 ,,	− 1·6
8 ,, — 9 ,,	56 ·50 ,,	5 ·20 ,,	− 0·8
9 ,, —10 ,,	55 ·65 ,,	4 ·35 ,,	− 0·9
10 ,, —11 ,,	54 ·85 ,,	3 ·55 ,,	− 0·7
11 ,, —Midnight	54 ·20 ,,	2 ·90 ,,	− 0·6
Midnight— 1 A.M.	53 ·65 ,,	2 ·35 ,,	− 0·5
1 A.M.— 2 ,,	52 ·90 ,,	1 ·60 ,,	− 1·0
2 ,, — 3 ,,	51 ·95 ,,	0 ·65 ,,	− 0·9
3 ,, — 4 ,,	51 ·40 ,,	0 ·10 ,,	− 0·2
4 ,, — 5 ,,	51 ·50 ,,	0 ·20 ,,	+ 0·4

* θ_0 = 51°·3 F.

From this Table we find

$$\frac{\theta - \theta_0}{d\theta} = 4{\cdot}290\,;$$

$$d\theta = -\ 0{\cdot}233\ (\theta - \theta_0).$$

The day temperatures give

TABLE XXXV.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	cos z.*
6 A.M.— 7 A.M.	53°·25 F.	1°·95 F.	+ 1·3	0·124
7 ,, — 8 ,,	54 ·85 ,,	3 ·55 ,,	+ 1·9	0·279
8 ,, — 9 ,,	57 ·05 ,,	5 ·75 ,,	+ 2·5	0·413
9 ,, —10 ,,	59 ·60 ,,	8 ·30 ,,	+ 2·6	0·527
10 ,, —11 ,,	61 ·90 ,,	10 ·60 ,,	+ 2·0	0·606
11 ,, —Noon.	63 ·60 ,,	12 ·30 ,,	+ 1·4	0·649
Noon — 1 P.M.	64 ·65 ,,	13 ·30 ,,	+ 0·7	0·649
1 P.M.— 2 ,,	65 ·00 ,,	13 ·70 ,,	± 0·0	0·606
2 ,, — 3 ,,	64 ·75 ,,	13 ·45 ,,	– 0·5	0·527
3 ,, — 4 ,,	63 ·95 ,,	12 ·65 ,,	– 1·1	0·413
4 ,, — 5 ,,	62 ·75 ,,	11 ·45 ,,	– 1·3	0·279
5 ,, — 6 ,,	61 ·25 ,,	9 ·95 ,,	– 1·7	0·124

Hence we have the following twelve equations:—

* $\cos z = 0{\cdot}0382 + 0{\cdot}622 \cos h$.

$$S_1 + R_{(53\cdot25)} = +\ 1\cdot3,$$
$$S_2 + R_{(54\cdot85)} = +\ 1\cdot9,$$
$$S_3 + R_{(57\cdot05)} = +\ 2\cdot5,$$
$$S_4 + R_{(59\cdot60)} = +\ 2\cdot6,$$
$$S_5 + R_{(61\cdot90)} = +\ 2\cdot0$$
$$S_6 + R_{(63\cdot60)} = +\ 1\cdot4,$$
$$S_6 + R_{(64\cdot65)} = +\ 0\cdot7,$$
$$S_5 + R_{(65\cdot00)} = \pm\ 0\cdot0,$$
$$S_4 + R_{(64\cdot75)} = -\ 0\cdot5,$$
$$S_3 + R_{(63\cdot95)} = -\ 1\cdot1,$$
$$S_2 + R_{(62\cdot75)} = -\ 1\cdot3,$$
$$S_1 + R_{(61\cdot25)} = -\ 1\cdot7.$$

From the Table of night Radiation, we know the first four Radiations, viz.:—

$$R_{(53\cdot25)} = -\ 0\cdot43, \qquad R_{(54\cdot85)} = -\ 0\cdot81,$$
$$R_{(57\cdot05)} = -\ 1\cdot33, \qquad R_{(59\cdot60)} = -\ 1\cdot95.$$

Hence, we find

$$S_1 = 1\cdot73, \qquad S_2 = 2\cdot71,$$
$$S_3 = 3\cdot83, \qquad S_4 = 4\cdot45.$$

Hence,

$$\begin{array}{rrl} a_1 = & 14\cdot00 & (S_1), \\ & 9\cdot71 & (S_2), \\ & 9\cdot28 & (S_3), \\ & 8\cdot45 & (S_4). \\ \hline & 10\cdot36. & \end{array}$$

From this we calculate

$$S_5 = 6\cdot28, \qquad S_6 = 6\cdot72.$$

By means of these values, we calculate the remaining eight Radiations, and construct the following Table of day Radiation :—

TABLE XXXVI.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
61°·25 F.	9°·95	– 3·45
61 ·90 ,,	10 ·60	– 4·28
62 ·75 ,,	11 ·45	– 4·01
63 ·60 ,,	12 ·30	– 5·32
63 ·95 ,,	12 ·65	– 4·93
64 ·65 ,,	13 ·30	– 6·02
64 ·75 ,,	13 ·45	– 4·95
65 ·00 ,,	13 ·70	– 6·28

These Radiations are plotted in Plate 9, and may be discussed in the following groups:—

No.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2·3)	10·67	– 3·91
(4·5)	12·47	– 5·12
(6·7·8)	13·48	– 5·75

From these data, I find the following most probable values :—

$$n = 1{\cdot}633, \qquad \log(p) = 1{\cdot}0788, \qquad p = 11{\cdot}99.$$

With these values I have constructed the curve *ab* in Plate 9, for comparison with the observed Radiations.

From the singular points of the curve we have

$$\frac{d\theta}{dt} = 0 < \begin{array}{l} \theta_0 = 51^\circ{\cdot}3 \text{ F.}, \\ ap \cos z = (\theta - \theta_0)^n \text{ at } 1^h\ 30^m \text{ P.M.} \end{array}$$

This gives the equation

$$10{\cdot}36 \times 11{\cdot}99 \times 0{\cdot}6132 = (13{\cdot}7)^n,$$

or,

$$n = 1{\cdot}655.$$

Plate 5, shows the derived curve, from which we find

$$\frac{d^2\theta}{dt^2} = 0 < \begin{array}{l} 9^h\ 30^m \text{ A.M.}, \\ 6^h\ 30^m \text{ P.M.}, \end{array}$$

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0{\cdot}609 = (8{\cdot}30)^{n-1} \times 2{\cdot}6,$$

$$\frac{a\beta p}{n} \times 0{\cdot}991 = (8{\cdot}15)^{n-1} \times 1{\cdot}9.$$

This gives the impossible value

$$n - 1 = -\ 43{\cdot}90.$$

This value of n is critical, as in the case of the February determination, because the temperatures, at the time of the forenoon and afternoon points of inflexion, approach equality.

10.—Month of October.

Table XXXVII.—*Mean Hourly Temperature.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	47°·8 F.	12	Noon.	55°·8 F.
1	1 A.M.	47 ·7 ,,	13	1 P.M.	56 ·2 ,,
2	2 ,,	47 ·3 ,,	14	2 ,,	55 ·6 ,,
3	3 ,,	47 ·1 ,,	15	3 ,,	54 ·4 ,,
4	4 ,,	46 ·9 ,,	16	4 ,,	53 ·5 ,,
5	5 ,,	46 ·9 ,,	17	5 ,,	52 ·4 ,,
6	6 ,,	47 ·2 ,,	18	6 ,,	51 ·5 ,,
7	7 ,,	47 ·9 ,,	19	7 ,,	50 ·7 ,,
8	8 ,,	49 ·1 ,,	20	8 ,,	50 ·0 ,,
9	9 ,,	50 ·7 ,,	21	9 ,,	49 ·4 ,,
10	10 ,,	52 ·7 ,,	22	10 ,,	48 ·8 ,,
11	11 ,,	54 ·5 ,,	23	11 ,,	48 ·3 ,,
Sunrise = 6^h 30^m A.M.			Sunset = 5^h 0^m P.M.		

This Table is plotted in Plate 2, and from it we find

Table XXXVIII.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
5 P.M.— 6 P.M.	51°·95F.	5·05	– 0·9
6 ,, — 7 ,,	51 ·10 ,,	4·20	– 0·8
7 ,, — 8 ,,	50 ·35 ,,	3·45	– 0·7
8 ,, — 9 ,,	49 ·70 ,,	2·80	– 0·6
9 ,, —10 ,,	49 ·10 ,,	2·20	– 0·6
10 ,, —11 ,,	48 ·55 ,,	1·65	– 0·5
11 ,, —Midnight.	48 ·05 ,,	1·15	– 0·5
Midnight — 1 A.M.	47 ·75 ,,	0·85	– 0·1
1 A.M.— 2 ,,	47 ·50 ,,	0·60	– 0·4
2 ,, — 3 ,,	47 ·20 ,,	0·30	– 0·2
3 ,, — 4 ,,	47 ·00 ,,	0·10	– 0·2
4 ,, — 5 ,,	46 ·90 ,,	0·00	– 0·0

* $\theta_0 = 46^\circ{\cdot}9$ F.

From this Table, we find

$$\frac{\theta - \theta_0}{d\theta} = 5{\cdot}611\,;$$

$$d\theta = -\ 0{\cdot}178\,(\theta - \theta_0).$$

The day temperatures give

TABLE XXXIX.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	cos z.*
7 A.M.— 8 A.M.	48°·50 F.	1·60	+ 1·2	0·114
8 ,, — 9 ,,	49 ·90 ,,	3·00	+ 1·6	0·252
9 ,, —10 ,,	51 ·70 ,,	4·80	+ 2·0	0·364
10 ,, —11 ,,	53 ·60 ,,	6·70	+ 1·8	0·444
11 ,, —Noon.	55 ·60 ,,	8·70	+ 1·3	0·485
Noon — 1 P.M.	56 ·00 ,,	9·10	+ 0·4	0·485
1 P.M.— 2 ,,	55 ·90 ,,	9·00	− 0·6	0·444
2 ,, — 3 ,,	55 ·00 ,,	8·10	− 1·2	0·364
3 ,, — 4 ,,	53 ·95 ,,	7·05	− 0·9	0·252
4 ,, — 5 ,,	52 ·95 ,,	6·05	− 1·1	0·114

* $\cos z = -\ 0{\cdot}119 + 0{\cdot}615 \cos h$.

Hence we have the following ten equations:—

$$
\begin{aligned}
S_1 + R_{(48\cdot 50)} &= +\,1\cdot 2,\\
S_2 + R_{(49\cdot 90)} &= +\,1\cdot 6,\\
S_3 + R_{(51\cdot 70)} &= +\,2\cdot 0,\\
S_4 + R_{(53\cdot 60)} &= +\,1\cdot 8,\\
S_5 + R_{(55\cdot 60)} &= +\,1\cdot 3,\\
S_5 + R_{(56\cdot 00)} &= +\,0\cdot 4,\\
S_4 + R_{(55\cdot 90)} &= -\,0\cdot 6,\\
S_3 + R_{(55\cdot 00)} &= -\,1\cdot 2,\\
S_2 + R_{(53\cdot 95)} &= -\,0\cdot 9,\\
S_1 + R_{(52\cdot 95)} &= -\,1\cdot 1.
\end{aligned}
$$

From the Table of night Radiation, or Plate 9, on which it is plotted, we know the value of the first three Radiations, viz.:—

$$R_{(48\cdot 50)} = -\,0\cdot 30, \qquad R_{(49\cdot 90)} = -\,0\cdot 57, \qquad R_{(51\cdot 70)} = -\,0\cdot 92.$$

Hence we find

$$S_1 = 1\cdot 50, \qquad S_2 = 2\cdot 17, \qquad S_3 = 2\cdot 92,$$

which, by the aid of the cosine column, give

$$
\begin{aligned}
a_1 = \;& 10\cdot 32 \quad (S_1),\\
& 8\cdot 61 \quad (S_2),\\
& 8\cdot 04 \quad (S_3).\\
\hline
& \mathbf{8\cdot 99.}
\end{aligned}
$$

Hence we have

$$S_4 = 3\cdot 99, \qquad S_5 = 4\cdot 36.$$

Solving the remaining seven equations, we obtain the following Radiations—

Table XL.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
52°·95 F.	6·05	− 2·60
53 ·60 ,,	6·70	− 3·06
53 ·95 ,,	7·05	− 3·07
55 ·00 ,,	8·10	− 4·12
55 ·60 ,,	8·70	− 3·06
55 ·90 ,,	9·00	− 4·59
56 ·00 ,,	9·10	− 3·96

These may be divided, as appears from Plate 9, into the following groups:—

No.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2·3)	6·60	− 2·91
(4·5)	8·40	− 3·59
(6·7)	9·05	− 4·275

Combining these groups, we find for the most probable values,

$$n = 1{\cdot}476, \qquad \log(p) = 0{\cdot}7784, \qquad p = 6{\cdot}0035.$$

From the singular points of the curve, we have

$$\frac{d^2\theta}{dt^2} = 0 < \begin{array}{l} \theta_0 = 46^\circ{\cdot}9 \text{ F.}; \\ ap \cos z = (\theta - \theta_0)^n \text{ at } 1^h \text{ P.M.}; \end{array}$$

or

$$8{\cdot}99 \times 6{\cdot}0035 \times 0{\cdot}475 = (9{\cdot}3)^n;$$

or

$$n = 1{\cdot}456.$$

Plate 5 shows the derived curve, from which we find

$$\frac{d^2\theta}{dt^2} = 0 < \begin{array}{l} 9^h\ 30^m \text{ A.M.}, \\ 3^h\ 30^m \text{ P.M.}, \end{array}$$

$$\frac{\alpha\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{\alpha\beta p}{n} \times 0{\cdot}609 = (3{\cdot}00)^{n-1} \times 2{\cdot}00,$$

$$\frac{\alpha\beta p}{n} \times 0{\cdot}793 = (7{\cdot}05)^{n-1} \times 1{\cdot}30.$$

Dividing, we find

$$n = 1{\cdot}813.$$

11.—Month of November.

Table XLI.—*Mean Hourly Temperature.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	40°·6 F.	12	Noon.	45°·4 F.
1	1 A.M.	40 ·5 ,,	13	1 P.M.	45 ·8 ,,
2	2 ,,	40 ·3 ,,	14	2 ,,	45 ·9 ,,
3	3 ,,	40 ·3 ,,	15	3 ,,	45 ·3 ,,
4	4 ,,	40 ·2 ,,	16	4 ,,	44 ·4 ,,
5	5 ,,	40 ·3 ,,	17	5 ,,	43 ·5 ,,
6	6 ,,	40 ·4 ,,	18	6 ,,	42 ·7 ,,
7	7 ,,	40 ·6 ,,	19	7 ,,	42 ·2 ,,
8	8 ,,	41 ·3 ,,	20	8 ,,	41 ·7 ,,
9	9 ,,	41 ·9 ,,	21	9 ,,	41 ·3 ,,
10	10 ,,	42 ·9 ,,	22	10 ,,	41 ·0 ,,
11	11 ,,	44 ·3 ,,	23	11 ,,	40 ·8 ,,
	Sunrise = $7^h\ 28^m$ A.M.			Sunset = $4^h\ 0^m$ P.M.	

This Table is plotted in Plate 2, and from it we find

Table XLII.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
4 P.M.— 5 P.M.	44°·85 F.	4·65	− 0·9
5 ,, — 6 ,,	43 ·20 ,,	3·00	− 0·8
6 ,, — 7 ,,	42 ·45 ,,	2·25	− 0·5
7 ,, — 8 ,,	41 ·95 ,,	1·75	− 0·5
8 ,, — 9 ,,	41 ·50 ,,	1·30	− 0·4
9 ,, —10 ,,	41 ·15 ,,	0·95	− 0·3
10 ,, —11 ,,	40 ·90 ,,	0·75	− 0·2
11 ,, —Midnight.	40 ·70 ,,	0·50	− 0·2
Midnight—1 A.M.	40 ·55 ,,	0·35	− 0·1
1 A.M.— 2 ,,	40 ·40 ,,	0·30	− 0·1
2 ,, — 3 ,,	40 ·30 ,,	0·10	± 0·0
3 ,, — 4 ,,	40 ·25 ,,	0·05	− 0·1
4 ,, — 5 ,,	40 ·25 ,,	0·05	+ 0·4
5 ,, — 6 ,,	40 ·35 ,,	0·15	+ 0·1
6 ,, — 7 ,,	40 ·50 ,,	0·30	+ 0·2

* θ_0 = 40°·2 F.

From this Table we find

$$\frac{\theta - \theta_0}{dt} = 5{\cdot}167\,;$$

$$d\theta = -\ 0{\cdot}194\,(\theta - \theta_0).$$

The day temperatures give

TABLE XLIII.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	$\cos z$.*
8 A.M.— 9 A.M.	41°·60 F.	1·40	+ 0·6	0·106
9 ,, —10 ,,	42 ·40 ,,	2·20	+ 1·0	0·214
10 ,, —11 ,,	43 ·60 ,,	3·40	+ 1·4	0·290
11 ,, —Noon.	44 ·85 ,,	4·65	+ 1·1	0·330
Noon — 1 P.M.	45 ·60 ,,	5·40	+ 0·4	0·330
1 P.M.— 2 ,,	45 ·85 ,,	5·65	+ 0·1	0·290
2 ,, — 3 ,,	45 ·60 ,,	5·40	− 0·6	0·214
3 ,, — 4 ,,	44 ·85 ,,	4·65	− 0·9	0·106

Hence, we have the following eight equations:—

* $\cos z = -\ 0{\cdot}250 + 0{\cdot}590 \cos h$.

$$S_1 + R_{(41\cdot60)} = +0\cdot6,$$

$$S_2 + R_{(42\cdot40)} = +1\cdot0,$$

$$S_3 + R_{(43\cdot60)} = +1\cdot4,$$

$$S_4 + R_{(44\cdot85)} = +1\cdot1,$$

$$S_4 + R_{(45\cdot60)} = +0\cdot4,$$

$$S_3 + R_{(45\cdot85)} = +0\cdot1,$$

$$S_2 + R_{(45\cdot60)} = -0\cdot6,$$

$$S_1 + R_{(44\cdot85)} = -0\cdot9.$$

Now the first four Radiations are known from the Table of Night Radiation, viz.:—

$$R_{(41\cdot60)} = -0\cdot27, \qquad R_{(42\cdot40)} = -0\cdot42,$$

$$R_{(43\cdot60)} = -0\cdot65, \qquad R_{(44\cdot85)} = -0\cdot90.$$

Hence we find,

$$S_1 = 0\cdot87, \qquad S_2 = 1\cdot42,$$

$$S_3 = 2\cdot05, \qquad S_4 = 2\cdot00;$$

From these values we find, by means of the cosine column,

$$\begin{aligned} a_1 = {} & 8\cdot89 \quad (S_1), \\ & 6\cdot64 \quad (S_2), \\ & 7\cdot07 \quad (S_3), \\ & 6\cdot06 \quad (S_4). \\ \hline & \mathbf{6\cdot975.} \end{aligned}$$

From this we construct the following Table and Plate 9.—

TABLE XLIV.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
44°·85 F.	4·65	− 0·90
44 ·85 ,,	4·65	− 1·67
45 ·60 ,,	5·40	− 1·60
45 ·60 ,,	5·40	− 2·02
45 ·85 ,,	5·65	− 1·95

Combining these in the most probable manner we have

No.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
(1·2)	44°·85 F.	4·65	− 1·285
(3·4)	45 ·60 ,,	5·40	− 1·810
(5)	45 ·85 ,,	5·65	− 1·950

Hence we find the most probable values

$$n = 2{\cdot}023, \qquad \log p = 1{\cdot}23222,$$

$$p = 17{\cdot}070.$$

Constructing now the derived curve, Plate 5, we have

$$\frac{d\theta}{dt} = 0 \begin{cases} \text{at } 4^{h}\ 0^{m}, \text{ A. M.,} \\ \text{at } 1^{h}\ 20^{m}, \text{ P. M.,} \end{cases}$$

$$\frac{d^2\theta}{dt^2} = 0 \begin{cases} \text{at } 10^{h}\ 30^{m}, \text{ A. M.,} \\ \text{at } 4^{h}\ 0^{m}, \text{ P. M.} \end{cases}$$

Hence we have

$$\theta_0 = 40^{\circ}{\cdot}2 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n;$$

or

$$6{\cdot}975 \times 17{\cdot}070 \times 0{\cdot}305 = (5{\cdot}7)^n,$$

$$n = 2{\cdot}183.$$

And

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{n} \times 0{\cdot}383 = (3{\cdot}40)^{n-1} \times 1{\cdot}4,$$

$$\frac{a\beta p}{n} \times 0{\cdot}866 = (4{\cdot}20)^{n-1} \times 0{\cdot}9;$$

or dividing,

$$n - 1 = 59{\cdot}5,$$

$$n = 60{\cdot}5;$$

an impossible value, due to the reasons already given.

12.—Month of December.

Table XLV.—*Mean Hourly Temperature.*

	Hour.	Temperature.		Hour.	Temperature.
0	Midnight.	38°·9 F.	12	Noon.	41°·9 F.
1	1 A.M.	38 ·8 ,,	13	1 P.M.	42 ·2 ,,
2	2 ,,	38 ·6 ,,	14	2 ,,	42 ·1 ,,
3	3 ,,	38 ·5 ,,	15	3 ,,	41 ·7 ,,
4	4 ,,	38 ·4 ,,	16	4 ,,	41 ·1 ,,
5	5 ,,	38 ·4 ,,	17	5 ,,	40 ·6 ,,
6	6 ,,	38 ·4 ,,	18	6 ,,	40 ·2 ,,
7	7 ,,	38 ·3 ,,	19	7 ,,	39 ·9 ,,
8	8 ,,	38 ·5 ,,	20	8 ,,	39 ·6 ,,
9	9 ,,	38 ·9 ,,	21	9 ,,	39 ·4 ,,
10	10 ,,	39 ·8 ,,	22	10 ,,	39 ·3 ,,
11	11 ,,	41 ·1 ,,	23	11 ,,	39 ·0 ,,
	Sunrise = 8^h 12^m A.M.		Sunset = 3^h 39^m P.M.		

This Table is plotted in Plate 2, and from it we find

Table XLVI.—*Night Radiation.*

Hour.	θ.	$\theta - \theta_0$.*	$\frac{d\theta}{dt}$.
4 P.M.— 5 P.M.	40°·85 F.	2·55	– 0·5
5 ,, — 6 ,,	40 ·40 ,,	2·10	– 0·4
6 ,, — 7 ,,	40 ·05 ,,	1·75	– 0·3
7 ,, — 8 ,,	39 ·75 ,,	1·45	– 0·3
8 ,, — 9 ,,	39 ·50 ,,	1·20	– 0·2
9 ,, —10 ,,	39 ·35 ,,	1·05	– 0·1
10 ,, —11 ,,	39 ·15 ,,	0·85	– 0·3
11 ,, —Midnight.	38 ·95 ,,	0·60	– 0·1

* θ_0 = 38°·3 F.

Hence we have

$$\frac{\theta - \theta_0}{d\theta} = 5{\cdot}100,$$

$$d\theta = -\ 0{\cdot}196\ (\theta - \theta_0).$$

The day temperatures give

TABLE XLVII.—*Day Temperatures.*

Hour.	θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.	cos z.*
9 A.M.—10 A.M.	39°·85 F.	1·55	+ 0·9	0·142
10 ,, —11 ,,	40 ·45 ,,	2·15	+ 1·3	0·215
11 ,, — Noon.	41 ·50 ,,	3·20	+ 0·8	0·252
Noon — 1 P.M.	42 ·05 ,,	3·75	+ 0·3	0·252
1 P.M.— 2 ,,	42 ·15 ,,	3·85	+ 0·1	0·215
2 ,, — 3 ,,	41 ·90 ,,	3·60	− 0·4	0·142

Hence we find the following six equations:—

$$S_1 + R_{(39\cdot85)} = +\ 0{\cdot}9,$$

$$S_2 + R_{(40\cdot45)} = +\ 1{\cdot}3,$$

$$S_3 + R_{(41\cdot50)} = +\ 0{\cdot}8,$$

$$S_3 + R_{(42\cdot05)} = +\ 0{\cdot}3,$$

$$S_2 + R_{(42\cdot15)} = -\ 0{\cdot}1,$$

$$S_1 + R_{(41\cdot90)} = -\ 0{\cdot}4.$$

* $\cos z = -\ 0{\cdot}310 + 0{\cdot}572 \cos h.$

From the preceding Table we have

$$R_{(39\cdot35)} = -0\cdot30, \qquad R_{(40\cdot45)} = -0\cdot42.$$

Hence we find

$$S_1 = 1\cdot20, \qquad S_2 = 1\cdot72.$$

These values give, by the column of cosines,

$$\begin{aligned} a_1 &= 8\cdot45 \quad (S_1), \\ &\ 8\cdot00 \quad (S_2). \\ &\overline{\ 8\cdot225.} \end{aligned}$$

Hence we find,

$$S_3 = 2\cdot07;$$

and finally,

TABLE LXVIII.—*Day Radiation.*

θ.	$\theta - \theta_0$.	$\frac{d\theta}{dt}$.
41°·50 F.	3·20	- 1·27
41 ·90 ,,	3·60	- 1·60
42 ·05 ,,	3·75	- 1·77
42 ·15 ,,	3·85	- 1·82

Combining all these observations together, we find the most probable values to be, including the six possible combinations,

$$n = 2\cdot035, \quad \log p = 0\cdot80791, \quad p = 6\cdot425.$$

Constructing the derived curve, Plate 5, we find

$$\frac{d\theta}{dt} = 0 < \begin{matrix} 7^h\ 0^m \text{ A.M.}, \\ 1^h\ 16^m \text{ P.M.}; \end{matrix}$$

$$\frac{d^2\theta}{dt^2} = 0 < \begin{matrix} 10^h\ 30^m \text{ A.M.}, \\ 3^h\ 30^m \text{ P.M.} \end{matrix}$$

We find

$$\theta_0 = 38°\cdot3 \text{ F.},$$

$$ap \cos z = (\theta - \theta_0)^n;$$

or

$$8\cdot225 \times 7\cdot26 \times 0\cdot231 = (3\cdot9)^n,$$

$$n = 1\cdot924.$$

and,

$$\frac{a\beta p}{n} \times \sin h = (\theta - \theta_0)^{n-1} \times (S + R),$$

$$\frac{a\beta p}{p} \times 0\cdot383 = (2\cdot15)^{n-1} \times 1\cdot3,$$

$$\frac{a\beta p}{n} \times 0\cdot793 = (3\cdot10)^{n-1} \times 0\cdot6;$$

or, dividing one by the other,

$$n - 1 = 4\cdot09,$$

$$n = 5\cdot09.$$

This value is clearly inadmissible.

CONCLUSION.

Let us now discuss the Annual Variations of the Sunheat and Radiation coefficients already found :—

(A).—*Solar-atmospheric Coefficient* $= a_1$.

ANNUAL VARIATION OF a_1.

1	January, ...	7·18	7	July, ...	8·13
2	February, . . .	7·24	8	August, ...	9·90
3	March, ...	7·27	9	September, . . .	10·36
4	April, ...	10·04	10	October, ...	8·99
5	May, ...	10·50	11	November, . . .	6·97
6	June, ...	12·15	12	December, . . .	8·22

This Table shows how variable, from month to month, is the Solar effect in penetrating the atmosphere. The Table represents the effect of the Sun (if placed in the zenith), in raising the thermometer at Greenwich during one hour.

If we group the months in threes we find

		Sunheat penetrating power $= a_1$.
(1).	January, February, March, ...	7·23 ;
(2).	April, May, June, ...	10·90 ;
(3).	July, August, September, ...	9·47 ;
(4).	October, November, December, ...	8·96.

The next coefficient we have to consider is θ_0.

(B).—*Minimum Diurnal Temperature which controls the Diurnal Radiation.*

ANNUAL VARIATION OF θ_0.

1	January, ...	37°·0 F.	7	July, ...	57°·3 F.
2	February, . . .	38 ·1 ,,	8	August, ...	56 ·1 ,,
3	March, ...	38 ·8 ,,	9	September, . . .	51 ·3 ,,
4	April, ...	42 ·0 ,,	10	October, ...	46 ·9 ,,
5	May, ...	47 ·7 ,,	11	November, . . .	40 ·2 ,,
6	June, ...	51 ·3 ,,	12	December, . . .	38 ·3 ,,

If this Table be reduced, by Fourier's method, to an annual and semi-annual periodic inequality, we have

$$\begin{aligned}\theta_0 = A_0 + A_1 \cos\phi + A_2 \cos 2\phi,\\ + B_1 \sin\phi + B_2 \sin 2\phi,\end{aligned}$$

which represents the observations very closely;

$$\begin{aligned}12A_0 &= S_1 + S_2 + \&\text{c.} \ldots S_{12},\\ A_0 &= 45^\circ{\cdot}42 \text{ F};\end{aligned}$$

$$\begin{aligned}6A_1 = (S_{12} - S_6),&\\ + (S_1 + S_{11} - S_5 - S_7) \times 0{\cdot}866,&\\ + (S_2 + S_{10} - S_4 - S_8) \times 0{\cdot}500;&\end{aligned}$$

$$\begin{aligned}6B_1 = (S_3 - S_9),&\\ + (S_2 + S_4 - S_8 - S_{10}) \times 0{\cdot}866,&\\ + (S_1 + S_5 - S_7 - S_{11}) \times 0{\cdot}500;&\end{aligned}$$

$$\begin{aligned}A_1 &= -\,7{\cdot}27,\\ B_1 &= -\,6{\cdot}45.\end{aligned}$$

If we calculate the annual variation only, we find—

	Month.	θ_0 calculated.	Difference of observed and calculated.
1	January, ...	35°·90 F.	+ 1°·10 F.
2	February, . . .	36 ·20 ,,	+ 1 ·90 ,,
3	March, ...	38 ·97 ,,	− 0 ·17 ,,
4	April, ...	43 ·49 ,,	− 1 ·49 ,,
5	May, ...	48 ·48 ,,	− 0 ·78 ,,
6	June, ...	52 ·69 ,,	− 1 ·39 ,,
7	July, ...	54 ·94 ,,	+ 2 ·36 ,,
8	August, ...	54 ·61 ,,	+ 1 ·49 ,,
9	September, . . .	51 ·87 ,,	− 0 ·57 ,,
10	October, ...	47 ·38 ,,	− 0 ·48 ,,
11	November, . . .	42 ·36 ,,	− 2 ·16 ,,
12	December, . . .	38 ·15 ,,	+ 0 ·15 ,,

The excess of the observed as compared with the calculated values is 7·00, and the defect is − 7·04 : and the first approximation, by means of an annual variation, is very exact.

We must next discuss the variation of n.

(C).—*Law of Variation of n.*

ANNUAL VARIATION OF n.

1	January, ...	2·036	7	July, ...	1·522
2	February, . . .	1·946	8	August, ...	1·467
3	March, ...	1·717	9	September, . . .	1·644
4	April, ...	1·352	10	October, ...	1·466
5	May, ...	1·538	11	November, . . .	2·103
6	June, ...	1·430	12	December, . . .	1·979

Treating the values of n like those of θ_0, by Fourier's Theorem, I find

$$A_0 = 1{\cdot}681;$$
$$A_1 = +\,0{\cdot}301;$$
$$B_1 = +\,0{\cdot}061.$$

From this formula I calculate the following Table, which may be regarded as smoothing down the errors of observation*:—

	Month.	n calculated.	Difference of observed and calculated.
1	January, ...	1·90	+ 0·066
2	February, . . .	1·87	+ 0·066
3	March, ...	1·85	− 0·023
4	April, ...	1·77	− 0·237*
5	May, ...	1·62	+ 0·088
6	June, ...	1·53	+ 0·050
7	July, ...	1·37	+ 0·112
8	August, ...	1·40	− 0·013
9	September, . . .	1·52	+ 0·024
10	October, ...	1·64	− 0·314*
11	November, . . .	1·81	+ 0·193
12	December, . . .	1·86	− 0·001

* The values of n for April and October are at variance with those of the other ten months.

The values of n for the winter months (November to March) are near each other, and the mean value is 1·9562.

The values of n for the summer months (May to September) are also near each other, and the mean value is 1·5162.

The mean value of n for April and October is 1·4045, or less than that of the winter or summer months.

This result may be not accidental, but a real characteristic of the two months that follow the equinoxes.

If we arrange the preceding Tables of θ_0 and n, calculated for an annual variation, in terms of months of similar temperatures, we have

	Months.	θ_0.	n.
1	January, ... February, ...	36°·045 F.	1·925
2	December, . . . March, ...	38 ·555 ,,	1·860
3	April, ... November, . . .	42 ·930 ,,	1·745
4	May, ... October, ...	47 ·935 ,,	1·615
5	June, ... September, . . .	51 ·880 ,,	1·500
6	July, ... August, ...	54 ·815 ,,	1·435

If this Table be plotted to scale, it is evident that the curve formed by θ_0, n, is *accurately* a straight line.

Or, taking the six observations to be represented by the equation

$$n = \theta_0 \tan \alpha + \beta,$$

we find, from the fifteen combinations possible,

1·2 . .	Tan α = − 0·0259
1·3 . .	,, − 0·0261
1·4 . .	,, − 0·0261
1·5 . .	,, − 0·0269
1·6 . .	,, − 0·0260
2·3 . .	,, − 0·0263
2·4 . .	,, − 0·0261
2·5 . .	,, − 0·0270
2·6 . .	,, − 0·0261
3·4 . .	,, − 0·0260
3·5 . .	,, − 0·0273
3·6 . .	,, − 0·0256
4·5 . .	,, − 0·0291
4·6 . .	,, − 0·0261
5·6 . .	,, − 0·0222

Hence, *very accurately*,

$$\tan \alpha = - 0{\cdot}02619,$$

$$\beta = + 2{\cdot}868.$$

(D).—*Law of Variation of p.*

The last coefficient remaining for discussion is the parameter p, which varied as follows:—

ANNUAL VARIATION OF p.

	Month.	p.		Month.	p.
1	January, ...	14·66	7	July, ...	9·40
2	February, . . .	12·80	8	August, ...	6·32
3	March, ...	12·68	9	September, . . .	11·99
4	April, ...	5·80	10	October, ...	6·00
5	May, ...	6·98	11	November, . . .	17·07
6	June, ...	6·40	12	December, . . .	6·42

At first sight these values of p appear, like those of a_1, to vary irregularly from month to month; but if the Table be discussed with reference to the accurately-ascertained law of the relation of n and θ_0, we can find some approach to a relation between p and θ_0.

In the following Table the first column of figures contains the value of n calculated as a function of θ_0 from the formula

$$n = - 0{\cdot}02619\ \theta_0 + 2{\cdot}868$$

already found; and the last column contains the values of p, calculated from the monthly *observations* of $\frac{d\theta}{dt}$, assuming the calculated values of n.

ANNUAL VARIATION OF p. RECALCULATED.

	Month.	n.	θ.	p.
1	January, . . .	1·90	37°·0 F.	12·09
2	February, . . .	1·87	38 ·1 ,,	11·02
3	March, ...	1·86	38 ·8 ,,	16·51
4	April, ...	1·77	42 ·0 ,,	13·63
5	May, ...	1·62	47 ·7 ,,	8·55
6	June, ...	1·53	51 ·3 ,,	7·59
7	July, ...	1·37	57 ·3 ,,	5·89
8	August, . . .	1·40	56 ·1 ,,	5·56
9	September, . .	1·52	51 ·3 ,,	10·41
10	October, . . .	1·64	46 ·9 ,,	9·08
11	November, . . .	1·81	40 ·2 ,,	13·48
12	December, . . .	1·86	38 ·3 ,,	11·20

If these observations be grouped in pairs (as before), according to the values of θ_0, we find

	Months.	θ_0.	p.
1	January, ... February, ...	37°·5 F.	11·54
2	December, . . . March, ...	38 ·5 ,,	13·85
3	April, ... November, . . .	41 ·1 ,,	13·55
4	May, ... October, ...	47 ·3 ,,	8·36
5	June, ... September, . . .	51 ·3 ,,	9·00
6	July, ... August, ...	56 ·7 ,,	5·72

The first three and last three groups are naturally related to each other, and the right line that represents the mean law of variation of p, in terms of temperature, is the line passing through the centres of gravity of the first and last group. This line is

$$p = -\,0·415\ \theta_0 + 29·18.$$

The foregoing calculations give the variations of the Sunheat and Radiation coefficients from month to month, the latter depending on θ_0; but it may be useful to give the mean values of n and p, depending upon the range $(\theta - \theta_0)$, for the whole year. This is done as follows:—

(E).—*Mean Radiation Coefficients for the whole Year.*

In the next Table I give the observed Radiations for each month in terms of the range.

In the following Table I calculate the mean Radiation of all the months in terms of the range. And then I calculate the most probable values of n and p for the whole year.

	37°·0 F.	38°·1 F.	38°·3 F.	38°·8 F.	40°·2 F.	42°·0 F.	46°·9 F.	47°·7 F.	51°·3 F.	51°·3 F.	56°·1 F.	57°·3 F.	
$\theta - \theta_0$.	January.	February.	December.	March.	November.	April.	October.	May.	June.	September.	August.	July.	Mean.
1°	0·17	0·28	0·18	0·23	0·18	0·27	0·19	0·28	0·26	0·22	0·29	0·34	0·2408
2°	0·33	0·57	0·39	0·53	0·44	0·52	0·46	0·54	0·52	0·46	0·58	0·69	0·5025
3°	0·63	0·91	0·86	0·87	0·69	0·84	0·60	0·93	0·90	0·65	0·93	0·99	0·8167
4°	1·10	1·19	..	1·20	0·86	1·11	0·78	1·16	1·15	0·92	1·17	1·39	1·0940
5°	1·82	1·88	..	1·48	1·54	1·40	1·78	1·38	1·38	1·14	1·53	1·60	1·5390
6°	..	2·62	..	1·72	2·18	1·58	2·35	2·18	1·83	1·39	2·27	1·85	1·9970
7°	..	..	..	2·24	..	2·13	2·94	2·76	2·52	1·70	2·87	2·33	2·4363
8°	..	..	..	2·83	..	2·88	3·58	3·36	3·04	1·88	3·50	2·89	2·9950
9°	..	..	..	3·45	..	3·38	4·27	4·04	3·61	2·56	4·17	3·50	3·6225
10°	..	..	..	4·13	..	3·96	..	4·73	4·14	3·57	4·86	4·17	4·2230
11°	..	..	..	..	..	4·42	..	5·46	4·80	4·13	5·60	4·80	4·8683
12°	..	..	..	..	..	5·00	..	6·37	5·44	4·80	6·43	5·55	5·5983
13°	..	..	..	..	..	5·48	..	7·10	6·11	5·52	7·36	6·27	6·3067
14°	..	..	..	..	..	6·20	..	8·02	6·80	6·26	7·20	6·92	6·9000
15°	..	..	..	..	..	6·82	..	8·75	7·49	..	..	..	7·6867
16°	..	..	..	..	..	..	..	..	8·20	..	..	..	8·2000
17°	..	..	..	..	..	..	..	..	8·96	..	..	..	8·9600
18°	..	..	..	..	..	..	..	..	9·89	..	..	..	9·8900

If we take the mean values of $\frac{d\theta}{dt}$ from the last column, and discuss the eighteen results, three by three, we have three equations from each group to determine the values of n and p, from which we form the following Table:—

No.	Group.	n.	p.
i.	1 . 2 . 3	1·124	4·199
ii.	4 . 5 . 6	1·462	6·830
iii.	7 . 8 . 9	1·581	8·915
iv.	10 . 11 . 12	1·549	8·399
v.	13 . 14 . 15	1·385	5·558
vi.	16 . 17 . 18	1·592	10·100

Of these groups, No. i. must be set aside, because for such small values of the range $(\theta - \theta_0)$ the curve differs little from a right line. The group, No. vi., may be retained, although it is less reliable than the other four, because it depends upon the observations of a single month (June).

The mean values of n and p for the whole year, derived from the last five groups, are

$$n = 1·5138,$$

$$p = 7·9604.$$

If we calculate the values of $\frac{d\theta}{dt}$ with these numbers, from the equation

$$(\theta - \theta_0)^n + p\frac{d\theta}{dt} = 0,$$

we can construct the following Table:—

RELATION BETWEEN RANGE AND RADIATION (MEAN OF ENTIRE YEAR).

$\theta - \theta_0$.	$-\frac{d\theta}{dt}$ (observed).	$-\frac{d\theta}{dt}$ (calculated).	Difference.	Error per cent.
4°	1·0940	1·0242	+ 0·0698	6·40
5°	1·5390	1·4412	+ 0·0978	6·35
6°	1·9970	1·8915	+ 0·1055	5·27
7°	2·4363	2·3891	+ 0·0472	1·93
8°	2·9950	2·9243	+ 0·0707	2·36
9°	3·6225	3·4933	+ 0·1292	3·57
10°	4·2230	4·1008	+ 0·1222	2·90
11°	4·8683	4·7373	+ 0·1310	2·70
12°	5·5983	5·4045	+ 0·1938	3·46
13°	6·3067	6·0105	+ 0·2962	4·70
14°	6·9000	6·8215	+ 0·0785	1·50
15°	7·6867	7·5735	+ 0·1130	1·47
16°	8·2000	8·3499	- 0·1499	1·82
17°	8·9600	9·1419	- 0·1819	2·03
18°	9·8900	9·9743	- 0·0843	0·85

This Table shows that the formula* represents the observations very well between the ranges

$$\theta - \theta_0 = 7° \text{ F, and } \theta - \theta_0 = 18° \text{ F.}$$

(F.)—*Total Heat received during each Month of the Year.*

Taking the mean temperature of each month as constant, the total heat received is equal to the total heat radiated; and from the calculation of the Radiation we can estimate the total heat actually received at Greenwich, as compared with what the Sun attempts to deliver.

* viz., $(\theta - \theta_0)^{1 \cdot 5138} + 7 \cdot 9604 \frac{d\theta}{dt} = 0.$

The following Table contains the data necessary for this purpose:—

RADIATION (MONTH BY MONTH).

Month.	θ.	θ_0.	n.	p.	$\theta - \theta_0$.	Radiation.
January, . . .	38°·9 F.	37°·0 F.	1·97	13·83	1·9	0·256
February, . . .	40 ·4 ,,	38 ·1 ,,	1·88	13·37	2·3	0·358
March, ...	42 ·8 ,,	38 ·8 ,,	1·74	13·08	4·0	0·853
April, ...	48 ·7 ,,	42 ·0 ,,	1·58	11·75	6·7	1·718
May, ...	54 ·4 ,,	47 ·7 ,,	1·45	9·38	6·7	1·680
June, ...	60 ·6 ,,	51 ·3 ,,	1·38	7·89	9·3	2·746
July, ...	63 ·9 ,,	57 ·3 ,,	1·39	5·40	6·6	2·547
August, . . .	62 ·6 ,,	56 ·1 ,,	1·48	5·90	6·5	2·706
September, . .	57 ·9 ,,	51 ·3 ,,	1·62	7·89	6·6	2·689
October, . . .	50 ·7 ,,	46 ·9 ,,	1·78	9·72	3·8	1·108
November, . . .	42 ·3 ,,	40 ·2 ,,	1·91	12·50	2·1	0·330
December, . . .	39 ·8 ,,	38 ·3 ,,	1·98	13·29	1·5	0·168

If we calculate, by Fourier's method, the quantities of heat received, we find

HEAT RECEIVED (MONTH BY MONTH).

	Observed.	Calculated.	Difference.
January, ...	0·256	0·050	+ 0·206
February, . . .	0·358	0·310	+ 0·048
March, ...	0·853	0·830	+ 0·023
April, ...	1·718	1·510	+ 0·208
May, ...	1·680	2·130	− 0·449
June, ...	2·746	2·660	+ 0·086
July, ...	2·542	2·870	− 0·328
August, ...	2·706	2·610	+ 0·096
September, . . .	2·689	2·090	+ 0·599
October, ...	1·108	1·410	− 0·302
November, . . .	0·330	0·790	− 0·460
December, . . .	0·168	0·260	− 0·092

The calculated curve crosses the mean line—

(1). *One* day after the Spring Equinox.

(2). *Twenty* days after the Autumn Equinox.

(3). Reaches a maximum
Twenty-four days after the Summer Solstice.

(4). Reaches a minimum
Twenty-four days after the Winter Solstice.

If we calculate for each month the quantities of heat received and radiated, as percentages of the maximum month, we find

HEAT RECEIVED (MONTH BY MONTH).

January, ...	1·74	July, ...	100·00
February, . . .	10·80	August, ...	91·10
March, ...	29·00	September, . . .	72·80
April, ...	52·60	October, ...	49·10
May, ...	74·20	November, . . .	27·50
June, ...	92·70	December, . . .	9·10

The theoretical quantities of heat received and radiated (if the atmosphere were removed) may be found by the formulæ given in Part I. of these Researches, and are as follows:—

HEAT RECEIVED, MONTH BY MONTH (WITHOUT ATMOSPHERE), LAT. 50°.
A.D. 1850 (PERCENTAGES).

16 January, . . .	21·4	16 July, ...	96·2
15 February, . .	35·5	15 August, . . .	83·1
17 March, . . .	55·9	14 September, . .	65·4
16 April, . . .	76·3	14 October, . . .	42·5
16 May, ...	92·5	13 November, . .	26·5
15 June, ...	100·0	13 December, . .	18·1

A comparison of these two Tables shows that the effect of the atmosphere in all cases diminishes the heat received from the Sun; but that (as might be anticipated) its effects are greatest in the winter months.

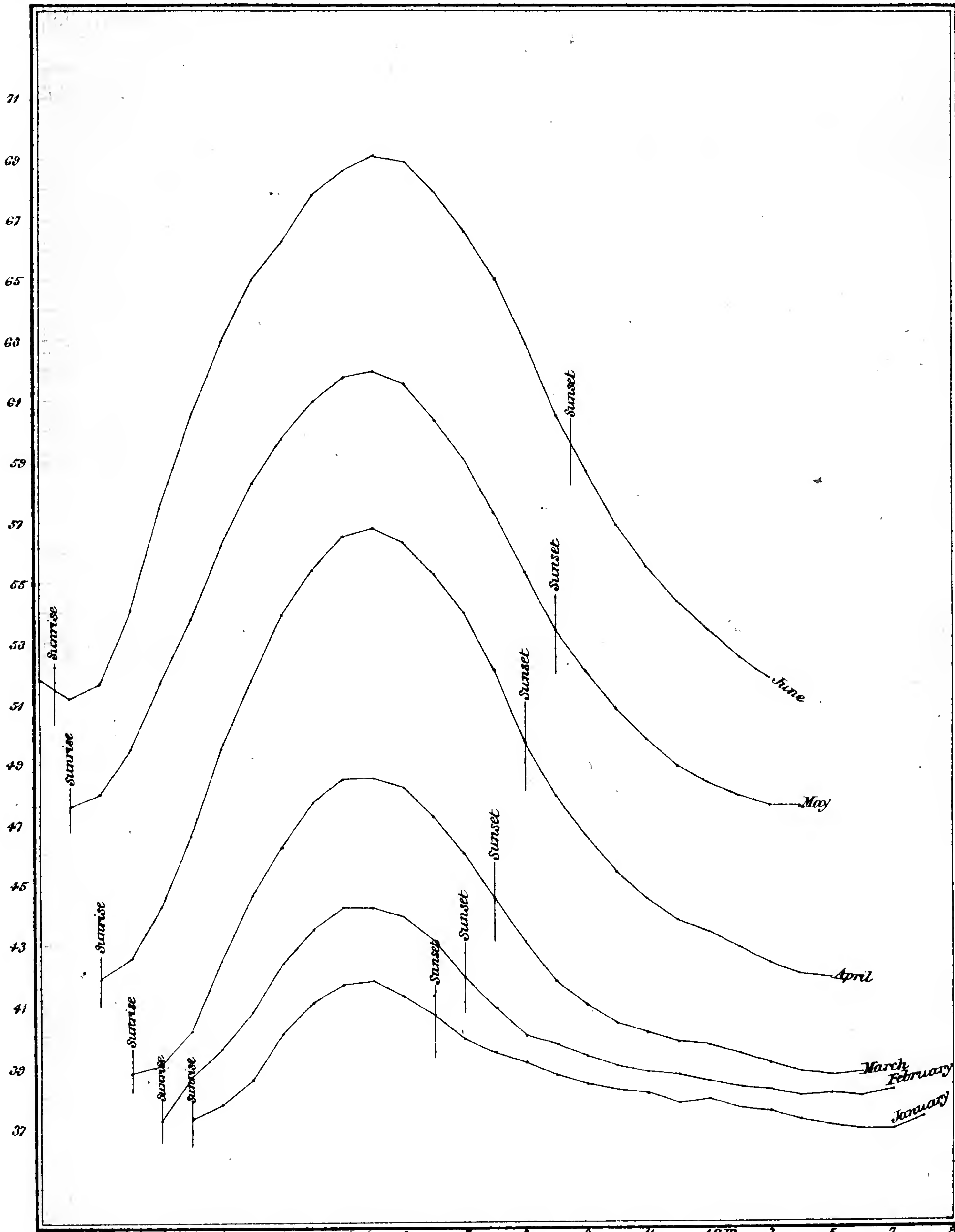
The average heat received in the four months—

$$\left.\begin{array}{l}\text{June,}\\ \text{July,}\\ \text{August,}\\ \text{September,}\end{array}\right\}\ \text{is } 2^{\circ}{\cdot}672 \text{ per hour.}$$

The average heat received in the four months—

$$\left.\begin{array}{l}\text{November,}\\ \text{December,}\\ \text{January,}\\ \text{February,}\end{array}\right\}\ \text{is } 0^{\circ}{\cdot}278 \text{ per hour,}$$

which is very little more than one-tenth of the former.



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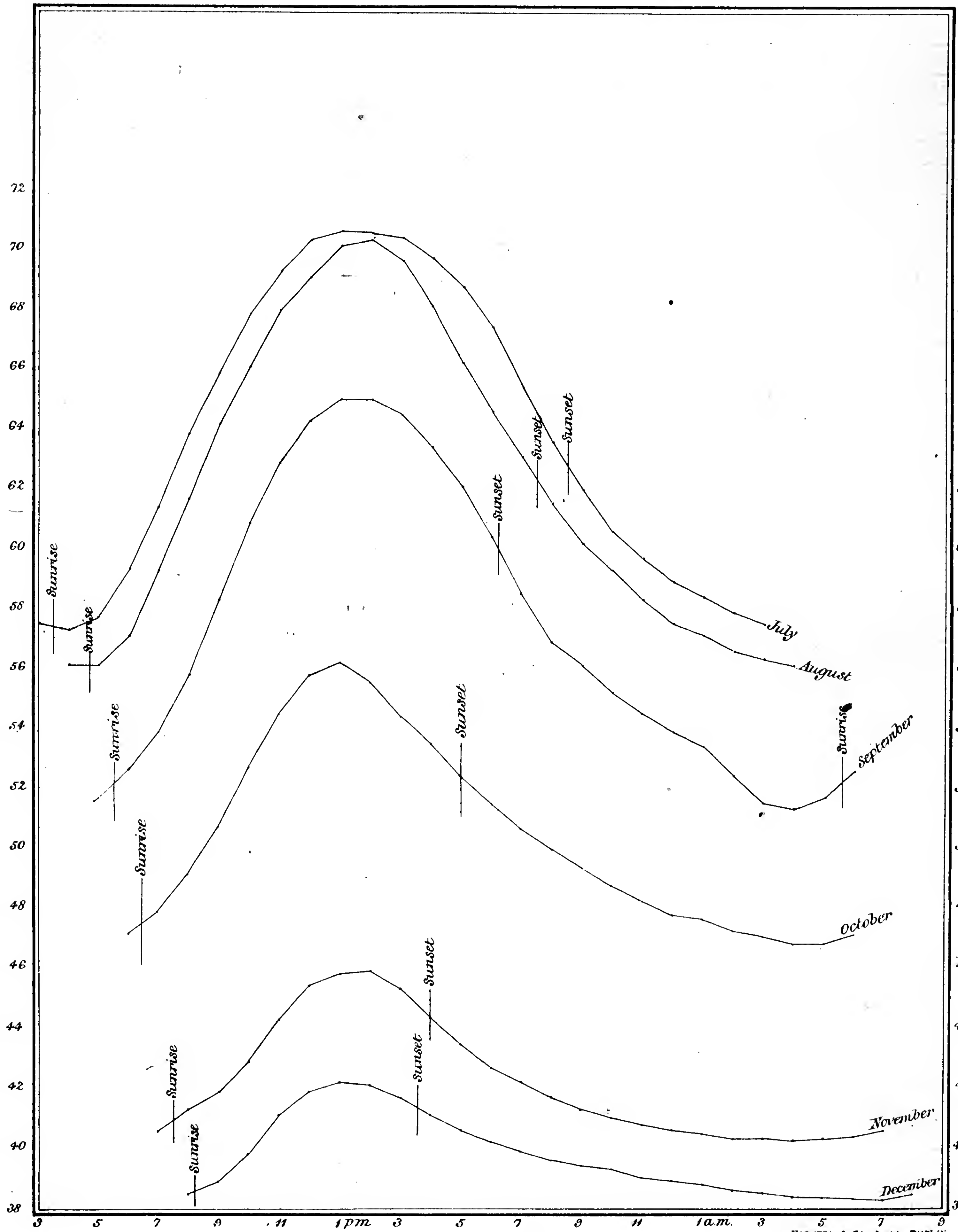
CURVE OF HOURLY TEMPERATURES. y-θ. x-t.



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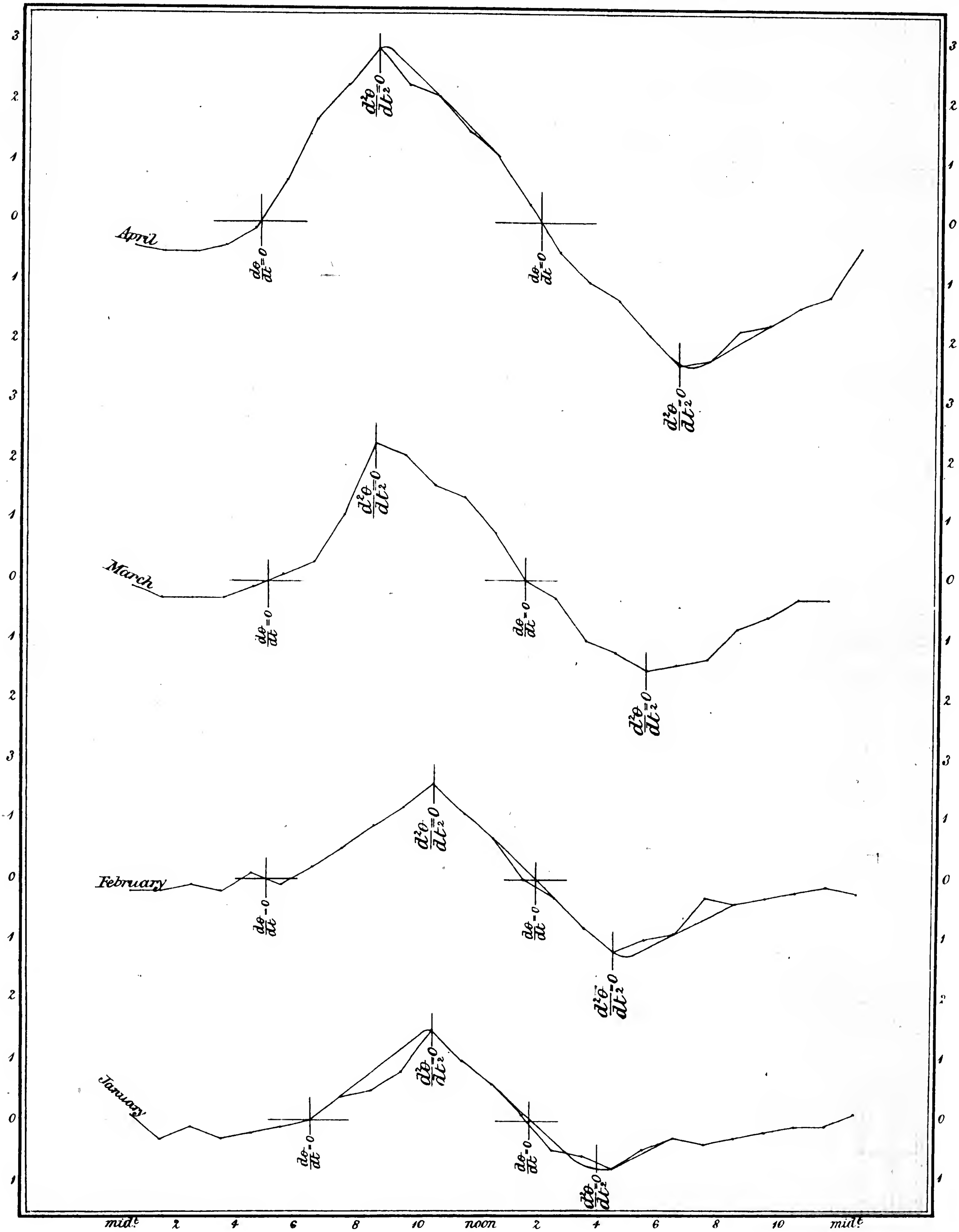

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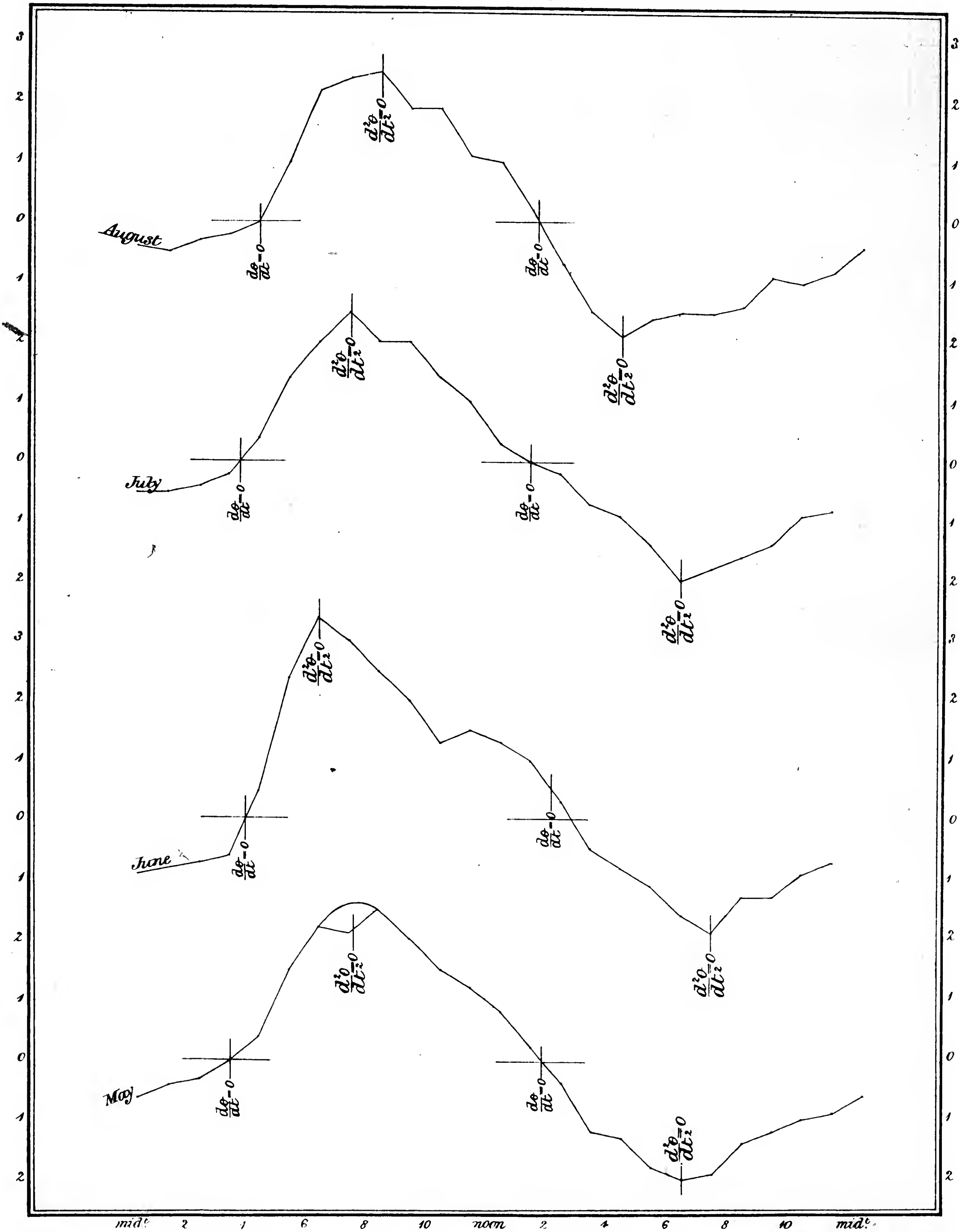
CURVE OF HOURLY CHANGE OF TEMPERATURES $y = \frac{d\theta}{dt}$. $x = t$.



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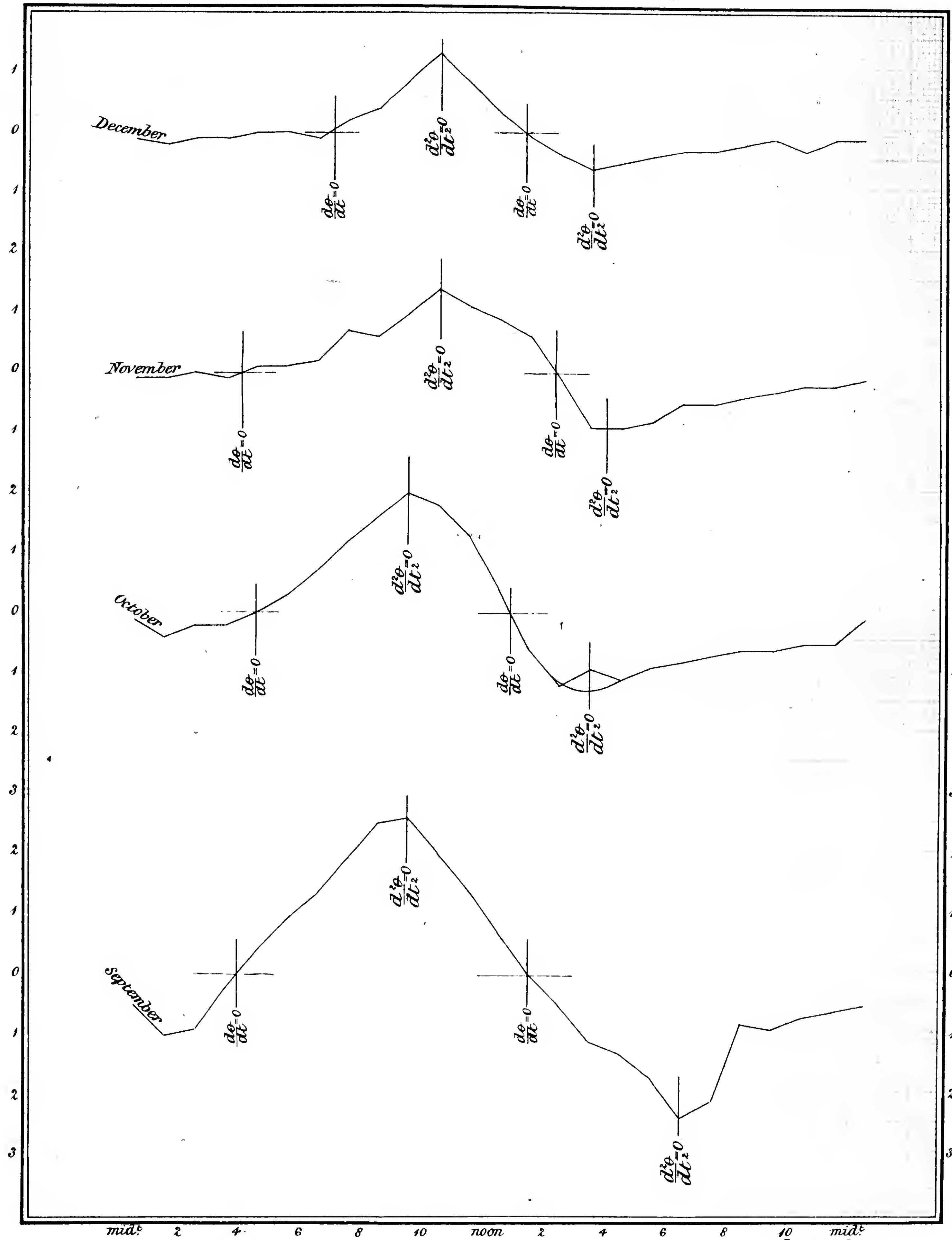

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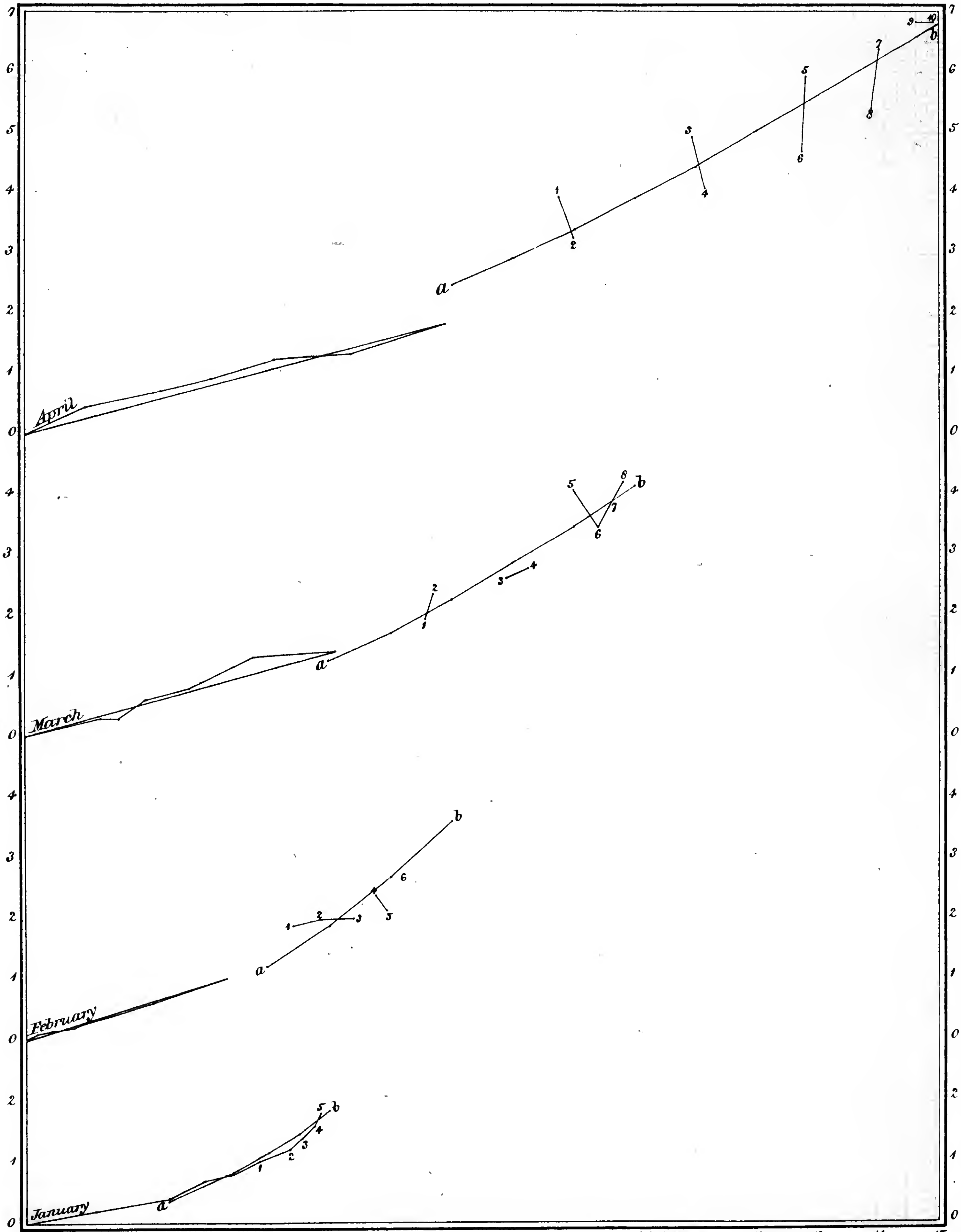

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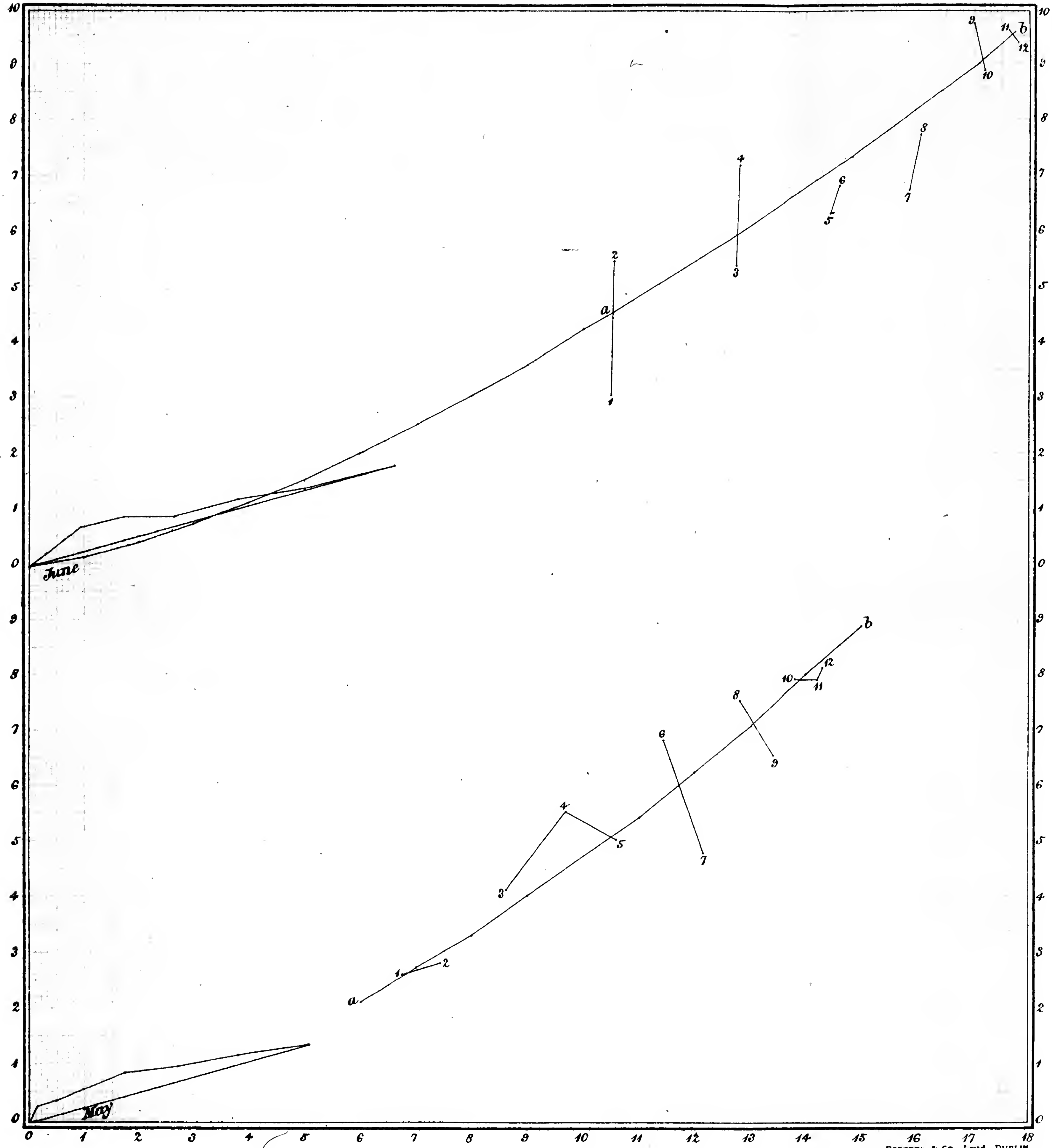
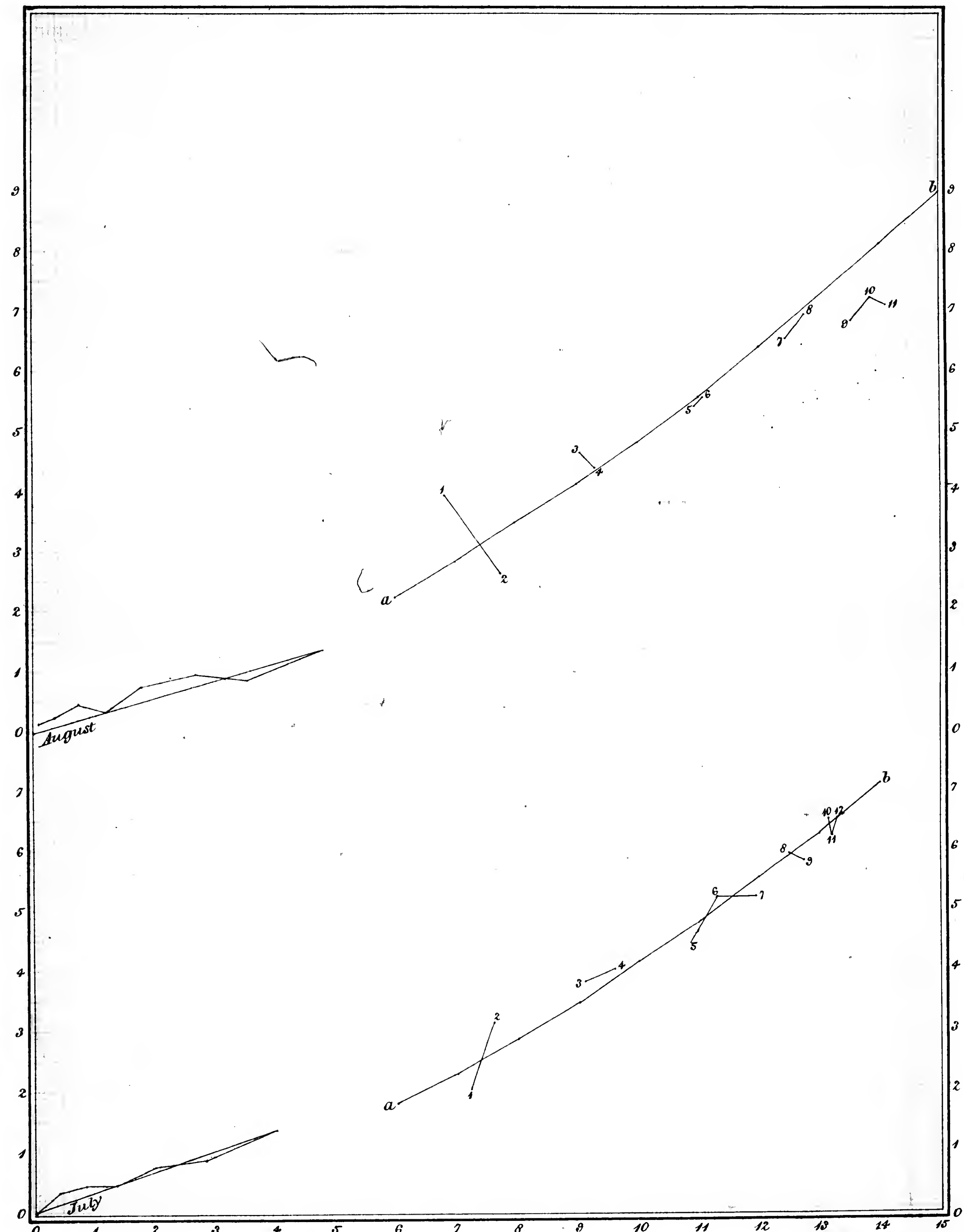
CURVE OF RADIATION, $y = \frac{d\theta}{\ }$ $x = \theta - \theta_0$.



CURVE OF RADIATION. $y = \frac{d\theta}{dt}$. $x = \theta - \theta_0$.



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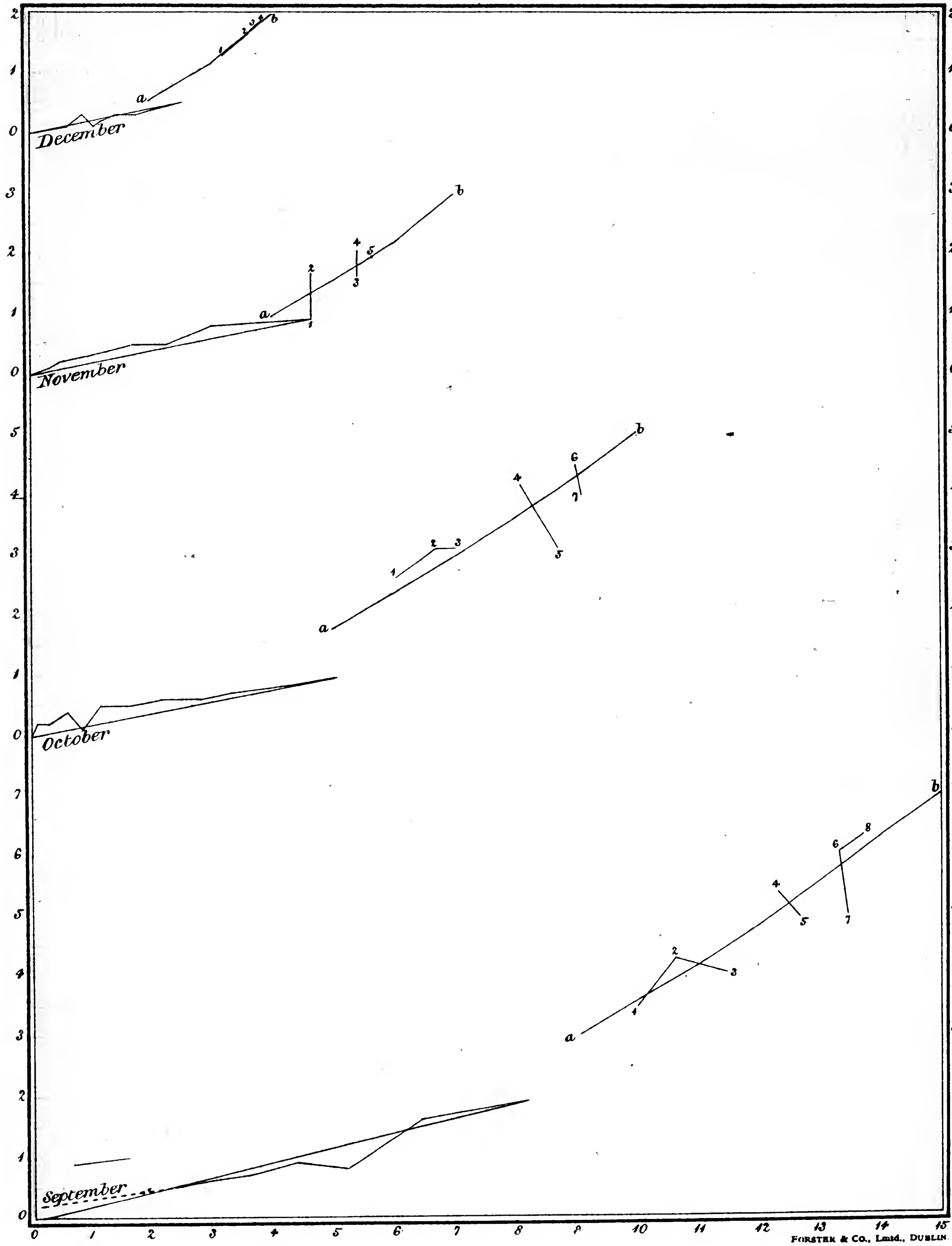

CURVE OF RADIATION. $y - \frac{d\theta}{dt}$. $x = \theta - \theta_0$.

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DYNAMICS AND MODERN GEOMETRY:

A NEW CHAPTER

IN THE

THEORY OF SCREWS.

BY

SIR ROBERT S. BALL, LL.D., F.R.S.,

Royal Astronomer of Ireland.

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No. IV.

Dynamics and Modern Geometry. A New Chapter in the Theory of Screws.

By Sir Robert S. Ball, LL.D., F.R.S., Royal Astronomer of Ireland.

[Read, December 13th, 1886.]

§ 1. **Preliminary.**—On the 9th April, 1883, I read before this Academy a Paper "On a Plane Representation of certain Dynamical Problems in the Theory of a Rigid Body." An addendum to this Paper was read on the 25th of June in the same year, and both were published in the *Proceedings*. (Ser. II., vol. IV., p. 29.[1])

The principles laid down in these two short Papers are, in the present Memoir, developed to an extent comparable with their geometrical and dynamical interest. The modern theory of the homographic division of a circle is found to be exactly adapted to the investigation of a series of dynamical problems of the most general type. The geometrical conceptions of the double points and the Pascal line are shown to have a useful dynamical significance. There is no artificial limitation of the dynamical

[1] I also refer to a Paper by Professor Mannheim, in the *Comptes rendus* for 2nd February, 1885, entitled, "Représentation plane relative aux deplacements d'une figure de forme invariable assujettre a quatre conditions." In this Paper the author shows how the plane representation which I had given can be deduced from the elegant geometrical theory which he had brought before the Academy of Sciences on several occasions.

conditions necessary to render the geometrical methods applicable. The only restrictions which we shall impose are thus stated :—

(1). The material system consists of a single rigid body.
(2). The body has two degrees of freedom.
(3). The investigation only applies while the body remains in or near to its original position.

It will be observed that the forces are generally unrestricted, though in some cases it will be found that they should belong to what is often known as a 'conservative' system. Nor is the nature of the constraints prescribed in any way, except that they shall so limit the body that any position it is capable of assuming can be specified by two co-ordinates. The third condition expresses the well-known limit to the operation of the Theory of Screws. The effect of this condition is to confine our inquiries to four departments, as follows:—1st. The kinematics of small movements; 2nd. The theory of equilibrium; 3rd. The theory of impulsive forces; 4th. The theory of small oscillations.

I refer to the *Theory of Screws*[1] for the demonstrations of the fundamental theorems that are required. I commence by reciting three laws which govern the displacements of a body which has two degrees of freedom.

(1). A body with two degrees of freedom can twist about all the screws of a singly infinite system.
(2). The axes of all these screws form the generators of a conoidal cubic surface, which is called the *cylindroid*, and of which the equation is

$$z(x^2 + y^2) - 2mxy = 0.$$

(3). Each screw has a specific pitch appropriate to its situation on the cylindroid.

§ 2. **Representation of the Cylindroid by a Circle.**—The essence of the present Memoir lies in the geometrical representation of a screw by a point. The series of screws which constitute the cylindroid correspond to,

[1] *Theory of Screws*, by R. S. Ball. Dublin : Hodges & Figgis, 1876.

or are represented by, a series of points in a plane. By choosing a particular type of correspondence we can represent the screws of the cylindroid by the points of a circle. The problems in this Memoir are studied by the aid of the corresponding circle. We are conducted to this circle by two methods, one of which shows that the relation of the circle to the surface is not merely casual or artificial. This method will form the subject of a future communication, and for the present let it suffice to choose a very simple process for the discovery of the circle, though it must be admitted that the method now used will hardly suggest to the reader those developments of anharmonic and homographic theories which will be found later on in this Paper.

It is shown (*Theory of Screws*, p. 15) that the positions of the several screws on the cylindroid may be concisely defined by the intersections of the pairs of planes,

$$y = x \tan \theta,$$

$$z = m \sin 2\theta.$$

In these equations, θ varies in correspondence with the several screws, while m is a parameter expressing the *size* of the cylindroid. In fact, the whole surface, except parts of the nodal line, is contained between two parallel planes, of which the perpendicular interval is $2m$.

The pitch of the screw corresponding to θ is expressed by

$$p = p_0 + m \cos 2\theta,$$

where p_0 is a constant.

Eliminating θ between the equations for z and p, we obtain

$$(p - p_0)^2 + z^2 = m^2.$$

Let p and z be regarded as the current co-ordinates of a point. Then the locus of this point is the circle which forms the foundation of the present Memoir.

Any point on this circle being given, then its co-ordinates p and z are completely determined: m is, of course, the radius, and p_0 is the distance of the centre from a certain axis. Thus $\sin 2\theta$ and $\cos 2\theta$, and, consequently, $\tan \theta$, are known. We therefore see that the position of a screw and its

pitch are completely determined when the corresponding point on the circle is known. To each point of the circle corresponds one screw on the cylindroid. To each screw on the cylindroid corresponds one point on the circle. This simple correspondence is the origin of the present theory.

§ 3. **The Axis of Pitch.**—Let T (fig. 1) be the origin. Then p_0 is the perpendicular distance ST of the centre S from the axis PT. The ordinate AP is the pitch of the screw, and the line PT may be called the *axis of pitch*. We have, accordingly, the following theorem:—

The pitch of any screw on the cylindroid is equal to the perpendicular let fall from the corresponding point on the axis of pitch.

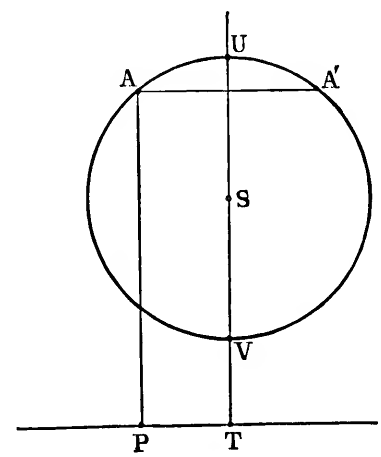

Fig. 1.

A parallel AA' to the axis of pitch cuts the circle in two points, A and A', which have equal pitch. The diameter perpendicular to the pitch axis intersects the circle in the point U of maximum and V of minimum pitch. These points, of course, correspond to the two principal screws on the cylindroid. The two screws of zero pitch are defined by the two real or imaginary points in which the axis of pitch cuts the circle.

A fundamental law of the pitch distribution on the several screws of a system is simply illustrated by this geometrical representation. The law states that if all the pitches be augmented by a constant addition, the pitches so modified will still be a possible distribution (*Theory of Screws*, pp. 19, 85). So far as the cylindroid is concerned, such a change would only mean a transference of the axis of pitch to some other parallel position. The diameter $2m$ merely expresses the size of the cylindroid, and is, of course, independent of the constant part in the expression of the pitch.

§ 4. **The Distance between two Screws.**—We shall often find it convenient to refer to a screw as simply equivalent to its corresponding point

on the circle. Thus, in fig. 2, the two points, A and B, may conveniently be called the screws A and B. The propriety of this language will be admitted when it is found that everything about a screw can be ascertained from the position of its corresponding point on the circle.

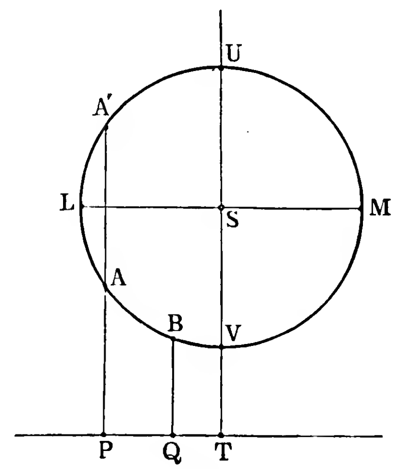

Fig. 2.

Let us, for instance, seek the shortest distance between the two screws A and B. Since all screws intersect the nodal axis of the cylindroid at right angles, the required shortest distance is simply the difference between the values of $m \sin 2\theta$ for the two screws: this is, of course, the difference of their abscissæ, *i. e.* the length PQ. Hence we have the following theorem :—

The shortest distance between two screws, A and B, is equal to the projection of the chord AB on the axis of pitch.

We thus see that every screw A on the cylindroid must be intersected by another screw A', and the chord AA' is, of course, perpendicular to the axis of pitch. A ray through S, parallel to the axis of pitch, will give two screws, L and M. These are the bounding screws of the cylindroid, and in each a pair of intersecting screws have become coincident. The two principal screws, U and V, lying on a diameter perpendicular to the axis of pitch, must also intersect.

If all the pitches be reduced by p_0, then the pitch axis passes through the centre of the circle, and the case assumes a simple type. The extremities of a chord perpendicular to the axis of pitch define screws of equal and opposite pitches, and every pair of such screws must intersect. The screws of zero pitch will then be the bounding screws, while the two principal screws will have pitches of $+ m$ and $- m$, respectively.

§ 5. **The Angle between two Screws.**—This important function also admits of a simple representation by the corresponding circle. Let A, B, as

before, denote the two screws; then, if θ and θ' be the angles corresponding to A and B (fig. 3),

$$AST = 2\theta; \quad BST = 2\theta',$$

whence $ASB = 2(\theta - \theta')$.

If H be any point on the circle, then

$$AHB = \theta - \theta',$$

whence we deduce the following theorem:—

The angle between two screws is equal to the angle subtended in the circle by their chord.

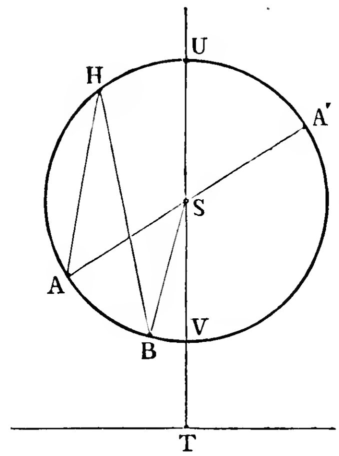

Fig. 3.

The extremities of a diameter denote a pair of screws at right angles: thus, A', in fig. 3, is the one screw on the cylindroid which is at right angles to A. The principal screws, U and V, are also seen to be at right angles.

The circular representation of the cylindroid is now complete. We see how the pitch of each screw is given, and how the perpendicular distance and the angle between every pair of screws can be concisely represented. We may therefore proceed to the dynamical theory, which gives to the cylindroid its interest, and we commence by proving a fundamental principle very analogous to an elementary theorem in Statics.

§ 6. **The Triangle of Twists.**—It is shown in the *Theory of Screws*, p. 17, that any three screws on the cylindroid possess the following property:—

If a body receive twists about these screws, so that the amplitude of each twist is proportional to the sine of the angle between the two non-corresponding screws, the body, after the last twist, will be restored to where it was before the first.

With the circular representation of the cylindroid we transform this theorem into the following:—

If any three screws, A, B, C (fig. 4), be taken on the circle, and if twists be applied to a body in succession, so that the amplitude of each twist

is proportional to the opposite side of the triangle ABC, then the body will be restored by the last twist to the place it had before the first.

From the analogy of wrenches, and of twist velocities to twists, we are also able to enunciate the following theorems:—

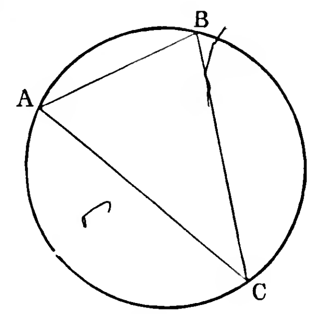

Fig. 4.

If wrenches upon the three screws A, B, C be applied to any rigid body, then these wrenches will equilibrate, provided that the intensity of each is proportional to the opposite side of the triangle.

If twist velocities about the three screws A, B, C animate a rigid body, then these twist velocities will neutralize if they are respectively proportional to the opposite sides of the triangle.

§ 7. **Decomposition of Twists and Wrenches.**—The theorems we have just enunciated lead to simple rules for effecting the composition, or the decomposition of twists or of wrenches. Let a twist on a screw X be given, and let it be required to find the components of this twist on any two given screws A, B, all three, of course, lying on the same cylindroid. Let ω be the amplitude of the twist on X. Then, by the last article, the following triad of twists on the screws X, A, B, respectively, will neutralize:—

$$\omega, \quad \omega\frac{BX}{AB}, \quad \omega\frac{AX}{AB},$$

whence the components on A and B of the twist about X are, so far as magnitudes and not signs are concerned,

$$\omega\frac{BX}{AB} \quad \text{and} \quad \omega\frac{AX}{AB}.$$

A similar proposition holds for wrenches.

§ 8. **Composition of Twists and Wrenches.**—Let two twists, of amplitudes α and β, about the screws A and B, respectively, be applied to a rigid body. It is required to find the single resultant screw X, and the amplitude ω

of the resulting twist. Divide AB (fig. 5) in the point I, so that the segments AI and BI shall be in the inverse ratio of α to β. Bisect the arc AB at H, and draw HI, which will cut the circle in the required point X.

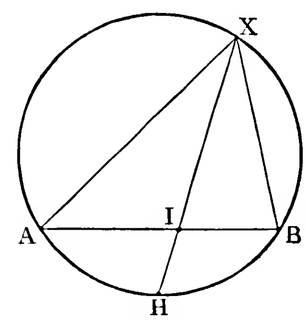

Fig. 5.

The value of ω is obtained from the equations,

$$\frac{\omega}{AB} = \frac{\alpha}{BX} = \frac{\beta}{AX}.$$

If the amplitudes α and β had opposite signs, then we should have taken for I the point dividing AB externally in the given ratio.

§ 9. **Screw Co-ordinates.**—We have developed, in the *Theory of Screws*, Chap. IV., the conception of Screw Co-ordinates. In the case of the cylindroid, the co-ordinates of any screw X, with respect to two standard screws A and B, are found by resolving a wrench of unit intensity on X into its two components on A and B. These components are said to be the co-ordinates of the screw. If we denote the co-ordinates of X by X_1 and X_2, we have

$$X_1 = \frac{BX}{AB}; \quad X_2 = \frac{AX}{AB}.$$

The co-ordinates satisfy the identical relation,

$$X_1^2 - 2X_1 X_2 \cos \epsilon + X_2^2 = 1,$$

where ϵ denotes the angle between the two screws of reference, that is, the angle subtended by the chord AB.

§ 10. **Reciprocal Screws.**—Perhaps the most striking relation between a pair of screws is exhibited when the two screws are *reciprocal.* The theory of reciprocal screws is of so much importance in Dynamics, that it becomes of interest to see what aspect it will assume in the circular representation. We might, indeed, anticipate that a relation of so much significance ought to be presented in a characteristic form, if the circular representation be a natural epitome of the theory. This anticipation is realized.

Two screws are reciprocal if a wrench on either does no work when a rigid body is twisted about the other. Every screw A on the cylindroid has one other reciprocal screw B lying also on the cylindroid. Denoting A and B by their corresponding points on the circle, we may enunciate the following theorem:—

The chord joining a pair of reciprocal screws passes through the pole of the axis of pitch.

We have shown (*Theory of Screws*, p. 21), that the condition that two screws shall be reciprocal is thus expressed:—

$$(p_\alpha + p_\beta) \cos \theta - d \sin \theta = 0,$$

where p_α and p_β are the pitches, where θ is the angle between the two screws, and d their shortest distance. We shall show that this condition is fulfilled, for any two screws A and B (fig. 6), whose chord passes through O, the pole of the axis of pitch PQ.

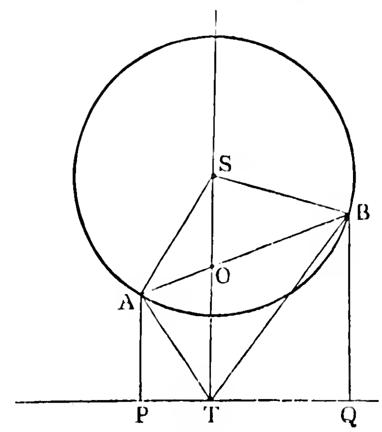

Fig. 6.

Since $SO \cdot ST = SA^2 = SB^2$, we have $\angle\, STA = \angle\, SAB$, and $\angle\, STB = \angle\, SBA$, whence $\angle\, ATB$ is bisected by ST, and therefore

$$\angle\, ATP = \tfrac{1}{2} \angle\, ASB = \theta = \angle\, BTQ.$$

It follows that $AP \cos \theta = PT \sin \theta$, since each is equal to the perpendicular from P on AT.

Similarly,

$$BQ \cos \theta = QT \sin \theta;$$

whence
$$(AP + BQ) \cos \theta - (PT + QT) \sin \theta = 0,$$

which, from §§ 4 and 5, reduces to

$$(p_\alpha + p_\beta) \cos \theta - d \sin \theta = 0.$$

The theorem has thus been proved.

We have, therefore, a very simple construction for finding the screw B

reciprocal to a given screw A. It is only necessary to join A to O, the pole of the axis of pitch, and the point in which this cuts the circle again gives B the required reciprocal screw.

We also notice that the two principal screws of the cylindroid are reciprocal, inasmuch as their chord contains O.

§ 11. **Another Geometrical Representation of the Pitch.**—We can obtain another geometrical expression for the pitch, which will be often more convenient than the perpendicular distance from the point to the axis of pitch.

Let A (fig. 7) be the point of which the pitch is required. Join AOB, let fall AP perpendicular on the axis of pitch PT, and produce AP to intersect BT at E. Then, since O is the pole of PT, the line PT bisects the angle ATE, and therefore AE must be bisected at P.

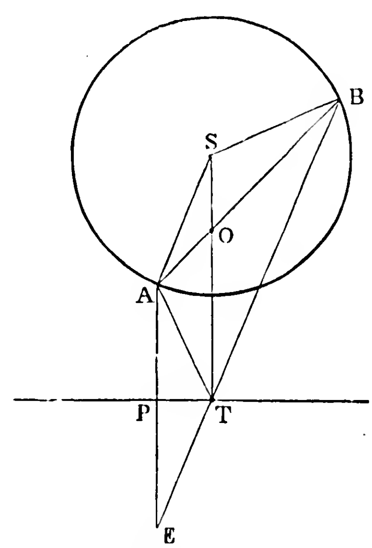

Fig. 7.

From similar triangles,

$$OB : AB :: OT : AE;$$

whence, if p_a be the pitch of A, and, of course, equal to AP, or $\frac{1}{2} AE$,

$$2p_a = \frac{AB \,.\, OT}{OB}.$$

But since the quadrilateral $ASBT$ is inscribable in a circle,

$$OT \,.\, OS = OA \,.\, OB;$$

whence, eliminating OT, we have, finally,

$$p_a = \frac{AO \,.\, AB}{2OS}:$$

as OS is constant, we see that p_a varies as $AO \,.\, AB$, whence the following theorem :—

If AB be any chord passing through O, the pole of the axis of pitch, then the pitch of the screw A is proportional to the product $AO \,.\, AB$.

§ 12. **Pitches of Reciprocal Screws.**—It is known that the sum of the reciprocals of the pitches of a pair of reciprocal screws on the cylindroid is constant. This is obvious from the geometrical representation. For, since the triangles APT and BQT (fig. 6) are similar, we have

$$AP : BQ :: TP : TQ :: OA : OB;$$

whence O is the centre of gravity of particles of masses $\frac{1}{p_\alpha}$ and $\frac{1}{p_\beta}$ placed at A and B, respectively.

From the known property of the centre of gravity,

$$AP \frac{1}{p_\alpha} + BQ \frac{1}{p_\beta} = OT \,.\left(\frac{1}{p_\alpha} + \frac{1}{p_\beta}\right);$$

but each of the terms on the left-hand side is unity, whence, as required,

$$\frac{1}{p_\alpha} + \frac{1}{p_\beta} = \frac{2}{OT}.$$

The second mode of representing the pitch also proves this theorem For since

$$p_\alpha = \frac{AO \,.\, AB}{2OS},$$

$$p_\beta = \frac{BO \,.\, BA}{2OS};$$

we have

$$p_\alpha + p_\beta = \frac{AB^2}{2OS}; \quad p_\alpha p_\beta = \frac{AB^2 . \, AO \,.\, BO}{4OS^2},$$

from which

$$\frac{1}{p_\alpha} + \frac{1}{p_\beta} = \frac{2 \,.\, OS}{OA \,.\, OB};$$

but $OA \,.\, OB$ is constant for every chord through O; and, as OS is constant, it follows that the sum of the reciprocals of the pitches must be constant.

§ 13. **The Virtual Coefficient.**—The function so designated is an important one in the Theory of Screws. It may be defined to be one-half of the expression

$$(p_\alpha + p_\beta) \cos \theta - d \sin \theta.$$

We have already seen that the evanescence of this function expresses that the two screws are reciprocal. In general this function possesses qualities which are analogous to those of the cosine of the angle between two rays. We shall obtain a geometrical expression.

Let A and B (fig. 8) be the two screws. Let, as usual, O be the pole of the axis of pitch PT, and let $P'T'$ be the polar of O'. From T let fall the perpendicular TF upon AT', and from O let fall the perpendicular OG upon AB.

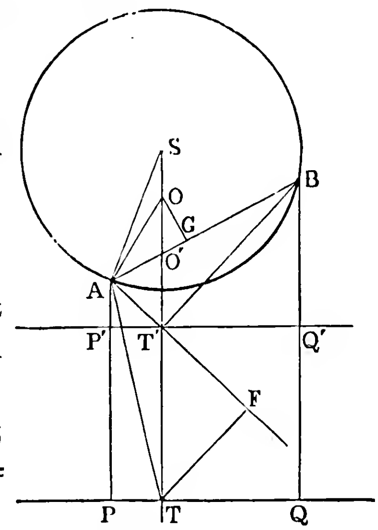

Fig. 8.

As before, we have $\angle AT'P' = \angle T'TF = \theta$; also, since $\angle SAO' = \angle AT'O'$, and $\angle SAO = \angle ATO$, we must have

$$\angle SAO' - \angle SAO = \angle AT'O' - \angle ATO, \quad \text{or} \quad \angle OAG = \angle TAF;$$

whence the triangles OAG and TAF are similar, and, consequently,

$$TF = OG \cdot \frac{AT}{AO} = OG \cdot \frac{AS}{OS};$$

but, as in § 10, we have

$$(p_\alpha - TT' + p_\beta - TT') \cos \theta - d \sin \theta = 0\,;$$

whence the virtual coefficient is simply,

$$TT' \cos \theta = OG \, \frac{AS}{OS},$$

and we have the following theorem :—

The virtual coefficient of any pair of screws varies as the perpendicular distance of their chord from the pole of the axis of pitch.

We also notice that the line TF expresses the actual value of the virtual coefficient.

The theorem of course includes, as a particular case, the property of reciprocal screws, which states that their chord passes through the pole of the pitch axis.

§ 14. **Another Investigation of the Value of the Virtual Coefficient.**—It will be instructive to investigate the theorem of the last article by a different part of the theory. We shall commence with a proposition in elementary geometry.

Let ABC (fig. 9) be a triangle circumscribed by a circle, the lengths of the sides being, as usual, a, b, c. Draw tangents at A, B, C, and thus form

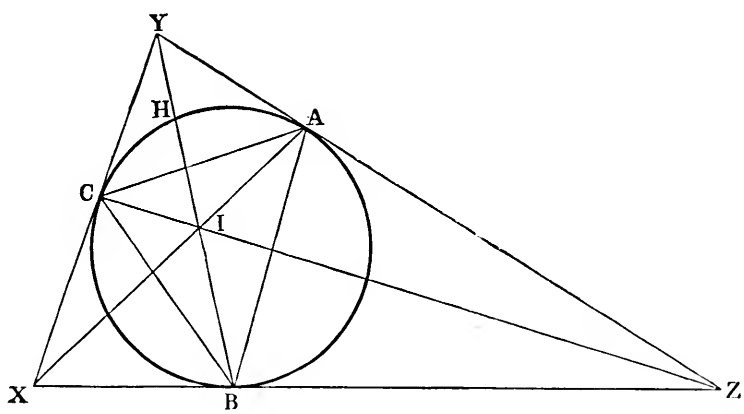

Fig. 9.

the triangle XYZ. It can be readily shown that if masses a^2, b^2, c^2 be placed at A, B, C, their centre of gravity must lie on the three lines AX, BY, CZ. These lines must therefore be concurrent at I, which is the centre of gravity.

Let BY intersect the circle again at H. Then, since AC is the polar of Y, the arc AC is divided harmonically at H and B; consequently the four points A, C, B, H subtend an harmonic pencil at any point on the circle. Let that point be B, then BC, BI, BA, BZ form a harmonic pencil; hence CZ is cut harmonically, and consequently Z must be the centre of gravity of particles, $+ a^2$ at A, $+ b^2$ at B, and $- c^2$ at C.

Suppose the axis of pitch to be drawn (it is not shown in the figure), and let h be the perpendicular let fall from Z on this axis, also let p_1, p_2, p_3 be the pitches of the screws A, B, C.

Then, by a familiar property of the centre of gravity, we must have

$$p_1 a^2 + p_2 b^2 - p_3 c^2 = (a^2 + b^2 - c^2)\, h = 2abh \cos C.$$

We shall take A, B as the two screws of reference, and if ρ_1 and ρ_2 be the co-ordinates of C with respect to A and B; then, from the principles of screw co-ordinates (*Theory of Screws*, p. 31), we have

$$p_3 = p_1 \rho_1^2 + p_2 \rho_2^2 + 2\varpi_{12} \rho_1 \rho_2,$$

where ϖ_{12} is the virtual coefficient of A and B. In the present case we have

$$\rho_1 = \frac{a}{c}; \quad \rho_2 = \frac{b}{c};$$

whence,

$$p_1 a_2^2 + p_2 b^2 - p_3 c^2 + 2\varpi_{12}\, ab = 0;$$

and, finally,

$$\varpi_{12} = -\, h \cos C.$$

The negative sign has no significance for our present purpose, and hence we have the following theorem :—

The virtual coefficient of two screws is equal to the cosine of the angle subtended by their chord, multiplied into the perpendicular from the pole of the chord on the axis of pitch.

This is, perhaps, the most concise geometrical expression for the virtual coefficient. It vanishes if the perpendicular becomes zero, for then the chord must pass through the pole of the pitch axis, and the two screws be reciprocal. The cosine enters the expression in order that its evanescence, when $P = 90$, may provide for the circumstance that the perpendicular is then infinite.

This result is easily shown to be equivalent to that of the last article by the following theorem :—

If any two chords be drawn in a circle, then the cosine of the angle subtended by the first chord, multiplied into the perpendicular distance from

its pole to the second chord, is equal to the cosine of the angle subtended by the second chord, multiplied into the perpendicular from its pole to the first chord.

It follows that the virtual coefficient must be equal to the perpendicular from the pole of the axis of pitch upon the chord joining the two screws, multiplied into the cosine of the angle in the arc cut off by the axis of pitch. This is the expression of § 13, namely,

$$OG\,\frac{AS}{OS}.$$

§ 15. **Application of Screw Co-ordinates.**—It will be useful to show how the geometrical form for the virtual coefficient is derived from the theory of screw co-ordinates. Let α_1, α_2, and β_1, β_2 be the co-ordinates of two screws on the cylindroid; then, if the screws of reference be reciprocal, the virtual coefficient is (*Theory of Screws*, p. 35),

$$p_1\alpha_1\beta_1 + p_2\alpha_2\beta_2.$$

Let A, B (fig. 10) be the screws of reference, and let C and C' be the two screws of which the virtual coefficient is required. Let PQ be the

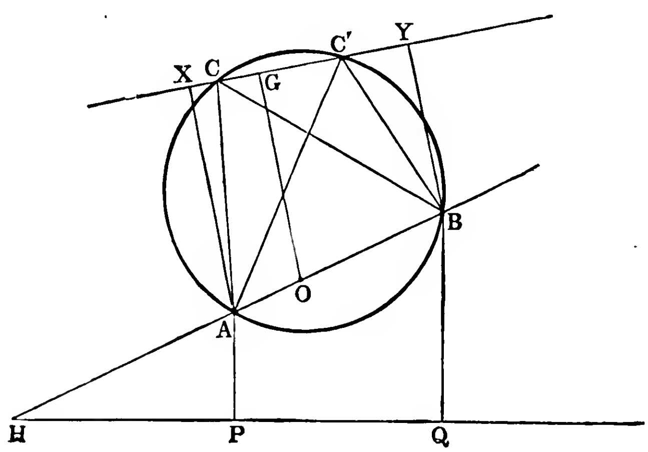

Fig. 10.

axis of pitch of which O is the pole, then O lies on AB, as the two screws of reference are reciprocal.

As AB is divided harmonically at O and H, we have

$$OA : OB :: HA : HB :: AP : BQ :: p_1 : p_2;$$

whence O is the centre of gravity of masses $\frac{1}{p_1}$, $\frac{1}{p_2}$ at A and B, respectively. If, therefore, AX, BY, OG be perpendiculars on CC', we have, from the principle of the centre of gravity,

$$\frac{1}{p_1} AX + \frac{1}{p_2} BY = \left(\frac{1}{p_1} + \frac{1}{p_2}\right) OG,$$

or,

$$p_2 AX + p_1 BY = (p_1 + p_2) OG;$$

but, by a well-known property of the circle, if m be the radius,

$$2m AX = AC \,.\, AC'; \quad 2m BY = BC \,.\, BC';$$

whence,

$$p_1 BC \,.\, BC' + p_2 AC \,.\, AC' = 2m(p_1 + p_2) OG = m \frac{OG \,.\, AB^2}{OS} \quad (\S\ 12),$$

or,

$$p_1 \frac{BC}{AB} \cdot \frac{BC'}{AB} + p_2 \frac{AC}{AB} \cdot \frac{AC'}{AB} = m \frac{OG}{OS}.$$

But, from the expressions for screw co-ordinates (§ 9), this reduces to

$$p_1 \alpha_1 \beta_1 + p_2 \alpha_2 \beta_2 = m \frac{OG}{OS}.$$

The required expression has thus been demonstrated.

§ 16. **Properties of the Virtual Coefficient.**—If the virtual coefficient be given, then the chord envelops a circle with its centre at the pole of the axis of pitch. Two screws can generally be found which have a given virtual coefficient with a given screw. If one screw be given, the screw which has the greatest virtual coefficient is found when the chord is perpendicular to the line joining the given screw to the pole of the axis of pitch.

Let A (fig. 11) be a given screw, and X a variable screw; then their virtual coefficient is proportional to OG, that is, to the sine of A, that is, to the length BX. Thus, as X varies round the circle, its virtual coefficient with the fixed screw A varies proportionally to the distance of X from the fixed point B.

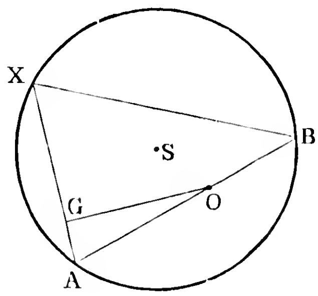

Fig. 11.

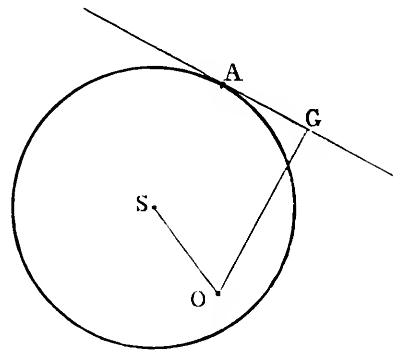

Fig. 12.

§ 17. **Another Construction for the Pitch.**—As the virtual coefficient of two coincident screws is equal to their pitch, we shall obtain another geometrical construction for the pitch by supposing two screws to coalesce. For (in fig. 12), let A be the chord joining the two coincident screws that is the tangent, then, from § 13, we have for the pitch,

$$m\,\frac{OG}{OS},$$

whence the following theorem :—

The pitch of any screw is proportional to the perpendicular on the tangent at the point let fall from the pole of the axis of pitch.

§ 18. **Another Proof of the Geometrical Expression for the Virtual Coefficient.**—We are now able to give an instantaneous proof of the theorem otherwise shown in §§ 13, 14, 15. If the two screws of reference be reciprocal, and if ρ_1 and ρ_2 be the co-ordinates of another screw, then it is known, from the theory of co-ordinates, that the virtual coefficients of this screw, with respect to the screws of reference, are $p_1\rho_1$ and $p_2\rho_2$, respectively (*Theory of Screws*, p. 35).

Thus (fig. 11), the virtual coefficient of X and A must be (§ 9),

$$p_1 \frac{BX}{AB};$$

but we know (§ 11),

$$p_1 = \frac{AO.AB}{2SO};$$

whence the virtual coefficient is

$$\frac{AO.BX}{2SO} = \frac{2m \sin A.AO}{2SO} = m\frac{OG}{OS},$$

as already determined. This is an instructive proof, besides being much shorter than the other methods.

§ 19. **Screws of Zero Pitch.**—A screw of zero pitch is reciprocal to itself. The tangent to a screw of zero pitch, inasmuch as it is the chord joining two reciprocal screws, must pass through the pole of the pitch line. This is, of course, the same thing as to say that the pitch line intersects the circle in two screws, each of which has zero pitch.

§ 20. **Consideration of a Special Case.**—We have, throughout this Paper, as in the *Theory of Screws* generally, referred only to the most general case of freedom of the second order. The circular representation, however, suggests one particular case which seems to have sufficient interest to demand a word of notice.

We have supposed that the axis of pitch occupies any arbitrary position. Let us now assume that it is a tangent to the representative circle. This really involves only a very slight specialization of the general case. It could be produced by augmenting the pitches of all the screws on the cylindroid, by such a constant as shall make one of the two principal screws have zero pitch.

The following properties of the screws are then obvious:—

1. There is only one screw of zero pitch, O.
2. The pitches of all the other screws have the same sign.
3. The maximum pitch is double the radius.

4. The screw O is reciprocal to every screw on the surface, and this is the only case in which a screw on the cylindroid is reciprocal to every other screw thereon.

We here illustrate a somewhat peculiar condition of the general theory of Reciprocal Screw systems. The general theory supposes that k screws determine one system, while $6-k$ reciprocal screws determine the reciprocal system. Every screw of either system is reciprocal to every screw of the other. The special case arises when one screw is found to be common to the two reciprocal systems; of course, as this screw must be reciprocal to itself, its pitch has to be zero.

§ 21. **The Kinetic Energy of Twisting Motion.**—Up to the present point of this Memoir we have been engaged with those parts of the theory which only involve the principles of Statics and Kinematics. We are now to show how the *kinetical* parts of the Theory of Screws can be studied and illustrated by the representative circle. It will be found that the geometrical theory of the circle lends itself with singular appropriateness to the elucidation of the theories of impulsive forces and of small oscillations.

We have studied (*Theory of Screws*, p. 55) the important parameter, which is expressed as u_θ, appropriate to each screw θ. This quantity is a linear magnitude, like the pitch; but, unlike the pitch, which is of course quite independent of the body twisted, the parameter u_θ depends on the mass and the position of the body. The significance of this parameter is best expressed by the following definition:—

If a rigid body of mass M twist about a screw θ, with the twist velocity $\dot{\theta}$, then the kinetic energy of the body may be written in the form

$$Mu_\theta^2\dot{\theta}^2,$$

where u_θ is a linear magnitude appropriate to the screw θ.

The function u_θ^2 is the arithmetic mean between the square of the radius of gyration and the square of the pitch. This is thus proved:—

Let $\eta_1, \ldots \eta_6$ be the co-ordinates of a screw of zero pitch on the same

axis as θ, reference being made to the absolute principal screws of inertia (*Theory of Screws*, p. 100), then the co-ordinates of θ are

$$\eta_1 + \frac{p_\theta}{4p_1} \cdot \frac{dR}{d\eta_1}, \quad \ldots \quad \eta_6 + \frac{p_\theta}{4p_6} \cdot \frac{dR}{d\eta_6}.$$

Substituting these values in the general expression for u_θ^2 (*Theory of Screws*, p. 55),

$$p_1^2\theta_1^2 + \ldots + p_6^2\theta_6^2,$$

we obtain

$$u_\theta^2 = u_\eta^2 + \frac{1}{2}p_\theta . \Sigma p_1\eta_1 . \frac{dR}{d\eta_1} + \frac{1}{16}p_\theta^2 \Sigma \left(\frac{dR}{d\eta_1}\right)^2;$$

but as shown (*loc. cit.*, p. 114),

$$\Sigma p_1 \eta_1 \frac{dR}{d\eta_1} = 0, \quad \Sigma \left(\frac{dR}{d\eta_1}\right)^2 = 8;$$

and if ρ_θ be the radius of gyration,

$$u_\eta^2 = \tfrac{1}{2}\rho_\theta^2,$$

whence, finally,

$$u_\theta^2 = \tfrac{1}{2}(\rho_\theta^2 + p_\theta^2),$$

which proves the theorem.

This result might have been obtained in a more simple manner, as follows:—

The kinetic energy of the body when twisting about θ may be regarded as the sum of two parts: one, the kinetic energy of the rotation; the other, of the translation. As the motion of each point of the body by the rotation is perpendicular to that of the translation, the total kinetic energy will be obtained by adding these two parts. The energy of the rotation is simply

$$\tfrac{1}{2}M\rho_\theta^2\dot{\theta}_2,$$

this being in accordance with the definition of the radius of gyration ρ_θ. The kinetic energy due to the translation is, of course,

$$\tfrac{1}{4}Mp_\theta^2\dot{\theta}^2,$$

whence the total kinetic energy is

$$\tfrac{1}{2}M\dot{\theta}^2(\rho_\theta^2 + p_\theta^2),$$

and therefore, as before,

$$u_\theta^2 = \tfrac{1}{2}(\rho_\theta^2 + p_\theta^2).$$

In the case where θ is one of the principal screws of inertia,

$$\rho_\theta^2 = p_\theta^2 = u_\theta^2.$$

We shall now examine the law of distribution of u_θ upon the several screws of the cylindroid. The representative circle will give a ready geometrical construction.

Let θ_1 and θ_2 be the two co-ordinates of θ relatively to any two screws of reference on the cylindroid. Then the components of the twist velocity will be $\dot{\theta}\theta_1$ and $\dot{\theta}\theta_2$. The actual velocity of any point of the body will necessarily be a linear function of these components. The square of the velocity will contain terms in which $\dot{\theta}^2$ is multiplied into θ_1^2, $\theta_1\theta_2$, θ_2^2, respectively. If, then, by integration we obtain the total kinetic energy, it must assume the form

$$M\dot{\theta}^2(\lambda\theta_1^2 + 2\mu\theta_1\theta_2 + \nu\theta_2^2),$$

whence, from the definition of u_θ,

$$u_\theta^2 = \lambda\theta_1^2 + 2\mu\theta_1\theta_2 + \nu\theta_2^2.$$

The three constants, λ, μ, ν, are the same for all screws on the cylindroid. They are determined by the material disposition of the body relatively to the surface.

We have chosen the two screws of reference arbitrarily, but this equation will receive a remarkable simplification when the two screws of reference have been chosen with special appropriateness.

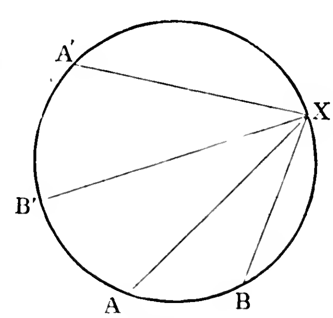

Fig. 13.

Let BX and AX (fig. 13) be denoted respectively by ρ_1 and ρ_2, and if ϵ denote the angle subtended by AB, we have (from § 9),

$$\lambda\rho_1^2 + 2\mu\rho_1\rho_2 + \nu\rho_2^2 - u_\theta^2(\rho_1^2 - 2\rho_1\rho_2\cos\epsilon + \rho_2^2) = 0.$$

Let us now transform this equation from the screws of reference A, B to another pair of screws A', B'. Let ρ_1', ρ_2' be the distances of X from A', B', respectively; then, from Ptolemy's theorem, we have the following equations :—

$$\rho_1 . A'B' = \rho_2' . AA' - \rho_1' . AB'.$$
$$\rho_2 . A'B' = \rho_2' . A'B - \rho_1' . BB',$$

We thus see that ρ_1 and ρ_2 are linear functions of ρ_1' and ρ_2', the several coefficients $A'B'$, $A'B$, &c., in these two equations being constant. The equation for u_θ^2 is thus to be transformed by a linear substitution for ρ_1 and ρ_2. Of course u_θ, being dependent only upon the position of X, is quite unaffected by the change of the screws of reference. We can therefore apply the well-known principle that the invariant of this binary quantic can only differ by a constant factor from the transformed value. The invariant is

$$(\lambda - u_\theta^2)(\nu - u_\theta^2) - (\mu + u_\theta^2 \cos \epsilon)^2.$$

This must be true for *every* point X, and therefore for all values of u_θ^2. It is necessary that the coefficients of the terms in the expression

$$u_\theta^4 \sin^2 \epsilon - u_\theta^2 (\lambda + \nu + 2\mu \cos \epsilon) + \lambda\nu - \mu^2$$

shall be severally proportional to those in the transformed expression

$$u_\theta^4 \sin^2 \epsilon' - u_\theta^2 (\lambda' + \nu' + 2\mu' \cos \epsilon') + \lambda'\nu' - \mu'^2.$$

We thus obtain the two equations of condition,

$$\frac{\sin^2 \epsilon'}{\sin^2 \epsilon} = \frac{\lambda' + \nu' + 2\mu' \cos \epsilon'}{\lambda + \nu + 2\mu \cos \epsilon} = \frac{\lambda'\nu' - \mu'^2}{\lambda\nu - \mu^2}.$$

The four quantities, λ', μ', ν', ϵ', may now be chosen arbitrarily, subject to these two equations, which are the necessary as well as the sufficient conditions. Indeed it is obvious that there must be but two independent quantities corresponding to the two positions of A' and B'.

We may impose two conditions on the four quantities, and for our present purpose we shall make

$$\lambda' = \nu'; \qquad \mu' = 0.$$

The equations of λ' and ϵ' are then

$$\frac{\sin^2 \epsilon'}{\sin^2 \epsilon} = \frac{2\lambda'}{\lambda + 2\mu \cos \epsilon + \nu} = \frac{\lambda'^2}{\lambda\nu - \mu^2},$$

and we obtain,

$$\lambda' = \frac{2(\lambda\nu - \mu^2)}{\lambda + 2\mu \cos \epsilon + \nu},$$

$$\sin^2 \epsilon' = \sin^2 \epsilon \frac{4(\lambda\nu - \mu^2)}{(\lambda + 2\mu \cos \epsilon + \nu)^2};$$

λ' is thus uniquely determined, and the expression for $\sin^2 \epsilon'$ gives for ϵ' four values of the type $\pm \epsilon'$, $\pm(\pi - \epsilon')$. The negative values are meaningless, and the two others are coincident, because the arc which subtends ϵ' on one side subtends $\pi - \epsilon'$ on the other.

There is thus a single pair of screws of reference which permit the expression for u_θ^2 to be exhibited in the canonical form

$$c^2 u_\theta^2 = \lambda' (\rho_1^2 + \rho_2^2).$$

We are now led to a simple geometrical representation for u_θ^2. Let A, B (fig. 14) be the two canonical screws of reference. Bisect AB in O', then

$$\begin{aligned} \rho_1^2 + \rho_2^2 &= BX^2 + AX^2, \\ &= 2AO'^2 + 2XO'^2, \\ &= 2XO' . YO' + 2XO'^2, \\ &= 2XY . XO'. \end{aligned}$$

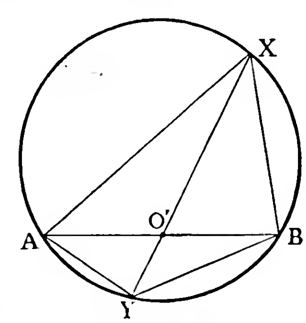

Fig. 14.

It is obvious that the point O' must have a critical importance in the kinetic theory, and its fundamental property, which has just been proved, is expressed in the following theorem :—

If a rigid body be twisting with the unit of twist velocity about any screw X on the cylindroid, then its kinetic energy is proportional to the rectangle XO', XY, where O' is a fixed point on the chord XY.

We are at once reminded of the theorem of § 11, in which a precisely

similar law is found for the distribution of pitch, only in this case another point, O, is used instead of the point O'. These two points, O and O', are both of great utility in the representative circle. From their analogy we obtain the following theorem, in which we call the polar of O' the *axis of inertia*:—

If a rigid body be twisting with the unit of twist velocity about X, then its kinetic energy is proportional to the perpendicular distance from X to the axis of inertia.

The geometrical construction for the pitch given in § 14 can also be applied to determine u_θ^2. This quantity is therefore proportional to the perpendicular from O' on the tangent at X.

It thus appears that the representative circle presents a vivid picture of the law of distribution of u_θ^2. The axis of inertia cannot cut the representative circle in real points, for otherwise we should have a twist velocity without any kinetic energy. There is no similar restriction to the axis of pitch. We thus see that O' must always lie inside the circle, but that O may be in any part of the plane.

§ 22. **Conjugate Screws of Inertia.**—We have made much use, in the *Theory of Screws*, of the conception of Conjugate Screws of Inertia. We shall here approach the subject in a different manner from that employed in the work referred to.

Let α be a screw about which a rigid body is twisting with a twist velocity $\dot{\alpha}$; let the body be simultaneously animated by a twist velocity $\dot{\beta}$ about a screw β. These two will compound into a twist velocity $\dot{\theta}$ about some screw θ. If the body only had the first twist velocity, its kinetic energy would be $Mu_\alpha^2\dot{\alpha}^2$. If it only had the second, the energy would be $Mu_\beta^2\dot{\beta}^2$. When it has both twist velocities together, the kinetic energy is $Mu_\theta^2\dot{\theta}^2$. Generally it will not be true that the resulting kinetic energy is equal to the sum of the components; but, under a special relation between α and β, we can have this equality; and when this is the case, then α and β are said to be *conjugate screws of inertia*. The necessary condition is thus expressed:—

$$u_\theta^2\,\dot{\theta}^2 = u_\alpha^2\,\dot{\alpha}^2 + u_\beta^2\,\dot{\beta}^2.$$

We have now to prove the following important theorem :—

Any chord through the pole of the axis of inertia intersects the circle in a pair of conjugate screws of inertia.

For we have,

$$\dot{\theta}^2 : \dot{\alpha}^2 : \dot{\beta}^2 :: AB^2 : BX^2 : AX^2;$$

but if AB passes through the pole of the axis of inertia, then the centre of gravity of masses $-AB^2$ at X, $+BX^2$ at A, and $+AX^2$ at B, will lie on the axis of inertia; and, accordingly,

$$AB^2u_\theta^2 = BX^2u_\alpha^2 + AX^2u_\beta^2;$$

whence,

$$\dot{\theta}^2u_\theta^2 = \dot{\alpha}^2u_\alpha^2 + \dot{\beta}^2u_\beta^2,$$

which proves the theorem.

Or we might have proceeded thus :—From Ptolemy's theorem (fig. 14),

$$AB \,.\, XY = AX \,.\, BY + AY \,.\, BX:$$

multiplying by $AB \,.\, XO'$,

$$AB^2 \,.\, XY \,.\, XO' = AX \,.\, AB \,.\, BY \,.\, XO' + AY \,.\, XO' \,.\, BX \,.\, AB;$$

but, from the property of the circle,

$$BY \,.\, XO' = AX \,.\, BO'; \quad AY \,.\, XO' = BX \,.\, AO';$$

whence

$$AB^2 \,.\, XY \,.\, XO' = AX^2 \,.\, AB \,.\, BO' + BX^2 \,.\, AB \,.\, AO',$$

from which we obtain, as before,

$$\dot{\theta}^2 u_\theta^2 = \dot{\alpha}^2 u_\alpha^2 + \dot{\beta}^2 u_\beta^2.$$

§ 23. **Impulsive Screws and Instantaneous Screws.**—A rigid body having two degrees of freedom lies initially at rest. It is suddenly acted upon by an impulsive wrench of large intensity acting for a short time. The body will, in general, commence to move by twisting about some screw on the cylindroid, and the kinetic problem now to be studied is the follow-

ing :—Given the impulsive screw, and the intensity of the impulsive wrench, find the instantaneous screw and the acquired twist velocity.

The problem will be rendered more concise by the conception of the reduced wrench (*Theory of Screws*, p. 60). It is to be remembered, that as the body is only partially free, there are an infinite number of screws on which wrenches would make the body commence to twist about a given screw on the cylindroid. For, let θ be an impulsive screw situated anywhere, and let an impulsive wrench on θ cause the body to commence to move by twisting about some screw, α, on the cylindroid. Let λ, μ, ν, ρ be any four screws reciprocal to the cylindroid. Then any wrench on a screw belonging to the system defined by these five screws will make the body commence to move by twisting about α. Let ϵ be that one screw on the cylindroid which is reciprocal to θ, then ϵ is reciprocal to the whole system defined by λ, μ, ν, ρ, θ, and, conversely, each screw of this system will be reciprocal to ϵ. We thus see that any screw, wherever situated, provided only that it is reciprocal to ϵ, will be an impulsive screw corresponding to α as an instantaneous screw. Any one of this system may, with perfect generality, be chosen as the impulsive screw. Among them there is one which has a special feature. It is that screw, ϕ, on the cylindroid which is reciprocal to η; and hence we have the following theorem :—

Given any screw, α, on the cylindroid, then there is another screw, ϕ, also on the cylindroid, such, that an impulsive wrench administered on ϕ will make the body twist about α.

This correspondence of the two systems of screws must be of the one-to-one type; for, suppose that two impulsive screws on the cylindroid had the same instantaneous screw, it would then be possible for two impulsive wrenches, of properly chosen intensities on two different screws, to produce equal and opposite twist velocities on the common instantaneous screw. The body would then not move, and therefore the two impulsive wrenches must equilibrate. But this is impossible, for they are on two different screws.

§ 24. **The Homographic Systems.**—From what has been shown it might

be expected that the points corresponding to the instantaneous screws and those corresponding to the impulsive screws should, on the representative circle, form two homographic systems. That this is so we shall now prove.

Let A, B (fig. 15) be a pair of impulsive screws, and let A', B' be respectively the corresponding pair of instantaneous screws, *i. e.* an impulsive wrench on A will make the body commence to twist about A', and similarly for B and B'. Let an impulsive wrench on A, of unit intensity, generate a twist velocity, α, about A', and let β be the similar quantity for B and B'.

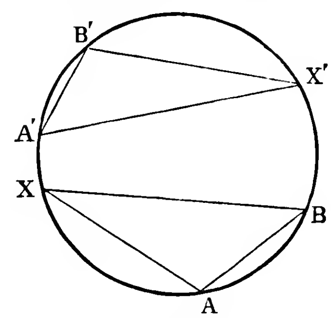

Fig. 15.

Let X be any other screw on which an impulsive wrench is to be applied to the body supposed quiescent. The body will commence to twist about some other screw, X', with a certain twist velocity ω. We can determine ω in the following manner:—The unit impulsive wrench on X can be replaced by two component wrenches on A and B, the intensities of these being

$$\frac{BX}{AB}, \frac{AX}{AB},$$

respectively.

These impulsive wrenches will severally generate twist velocities about A', B' of the value,

$$\alpha\frac{BX}{AB}, \beta\frac{AX}{AB}:$$

these components must, when compounded, produce the twist velocity ω about X', and, accordingly, we have

$$\alpha\frac{BX}{AB} = \omega\frac{B'X'}{A'B'}; \quad \beta\frac{AX}{AB} = \omega\frac{A'X'}{A'B'}.$$

Retaining A, B, A', B', as before, let us now introduce a second pair of points, Y and Y', instead of X and X', and writing ω' instead of ω, we have

$$\alpha\frac{BY}{AB} = \omega'\frac{B'Y'}{A'B'}; \quad \beta\frac{AY}{AB} = \omega'\frac{A'Y'}{A'B'};$$

whence, eliminating α, β, ω, ω', we have

$$\frac{BX}{AX} : \frac{BY}{AY} :: \frac{B'X'}{A'X'} \cdot \frac{B'Y'}{A'Y'}.$$

As the length of a chord is proportional to the sine of the subtended angle, we see that the anharmonic ratio of the pencil, subtended by the four points, A, B, X, Y at a point on the circumference, is equal to that subtended by their four correspondents, A', B', X', Y'. We thus learn the following important theorem :—

A system of points on the representative circle, regarded as impulsive screws, and the corresponding system of instantaneous screws, form two homographic systems.

§ 25. **The Homographic Axis.**—Let A, B, C, D (fig. 16) represent four impulsive screws, and let A', B', C', D' be the four corresponding instantaneous screws. Then, by the well-known homographic properties of the circle, the three points, L, M, N, will be collinear, and we have the following theorem :—

If A and B be any two impulsive screws, and if A' and B' be the corresponding instantaneous screws, then the chords AB' and BA' will always intersect upon the fixed right line XY.

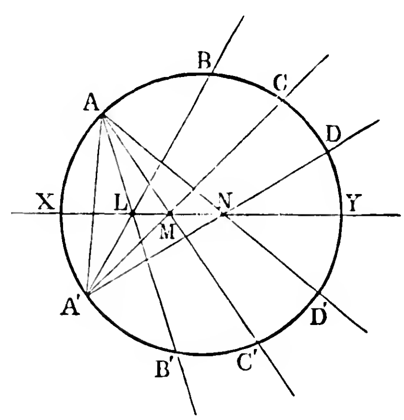

Fig. 16.

This right line is called the homographic axis. It intersects the circle in two points, X and Y, which are the double points of the homographic systems. These points enjoy a special dynamical significance. They are the two Principal Screws of Inertia, and hence—

The homographic axis intersects the circle in two points, each of which possesses the property, that an impulsive wrench administered on that screw will make the body commence to move by twisting about the same screw.

The method by which we have been conducted to the Principal Screws

of Inertia shows how it comes that there are in general two, and only two, of these screws on the cylindroid. The homographic axis is often known as the Pascal line, and thus we have a dynamical significance for Pascal's theorem.

§ 26. **Determination of the Homographic Axis.**—It is one of the fundamental parts of the theory of screws, that the two principal screws of inertia must be reciprocal, and must also be conjugate screws of inertia (*Theory of Screws*, p. 50). The homographic axis must therefore comply with the conditions thus prescribed. We have already shown (§ 10) the condition that two screws be reciprocal, and (§ 22) the condition that two screws be conjugate screws of inertia, and, accordingly, we see—

1°. That the homographic axis must pass through O, the pole of the axis of pitch.

2°. That the homographic axis must pass through O', the pole of the axis of inertia.

The point O and O' have been already discussed, and thus we have, as the simplest construction for the homographic axis, the chord joining O and O'.

§ 27. **Construction for Instantaneous Screws.**—The points O and O' afford a simple construction for the instantaneous screw, corresponding to a given impulsive screw. The construction depends upon the following theorem (*Theory of Screws*, p. 48):—

If two conjugate screws of inertia be regarded as instantaneous screws, then the impulsive screw corresponding to either is reciprocal to the other.

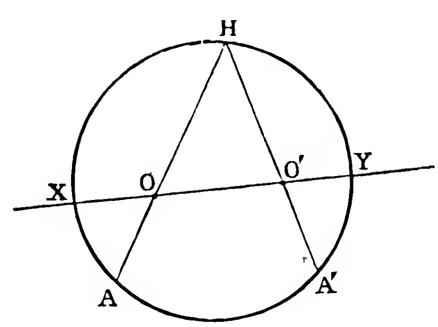

Fig. 17.

Let A be an impulsive screw (fig. 17); if we join AO we obtain H, the screw reciprocal to A; and if we join HO' we obtain A', the conjugate screw of inertia to H. Now, as A is the only screw reciprocal to H, it is necessary, by

the theorem just given, that an impulsive wrench on A must make the body commence to move by twisting about A'.

As O and O' are fixed, it follows from a well-known geometrical theorem, that the corresponding positions of A and A' form two homographic systems.

§ 28. **Twist Velocity Acquired by an Impulse.**—We can obtain a geometrical expression for the twist velocity acquired about A' by a unit impulsive wrench on A. We have proved this theorem in one of the earlier Papers already referred to; but the following demonstration is simpler than that there given.

It appears, from the *Theory of Screws*, p. 56, that the twist velocity acquired on α by an impulsive wrench on η, is proportional to

$$\frac{\varpi_{\eta\alpha}}{u_\alpha^2},$$

the numerator being the virtual coefficient is proportional to $AO.A'H$ (fig. 17), and as u_α^2 is proportional to $A'O'.A'H$, we see that the required ratio varies as

$$\frac{AO}{A'O'} \propto \frac{HO'}{HO};$$

hence we obtain the following theorem :—

The impulsive wrench on A, of an intensity proportional to HO, generates a twist motion about A', with a velocity proportional to HO'.

The problem of the effect of impulsive forces is thus completely solved, both as regards the instantaneous screw, and the instantaneous twist velocity acquired.

§ 29. **Another Construction for the Unit Velocity.**—A still more concise method of determining the instantaneous screw can be obtained if we discard the points O and O', and introduce a new fixed point, Ω, also on the homographic axis.

Let X, Y (fig. 18) be the two principal screws of inertia. Let A be an impulsive screw, and A' the corresponding instantaneous screw. Draw

through A the line AH parallel to XY. Join HA', and produce it to meet the homographic axis at Ω. Let α be the twist velocity generated by an

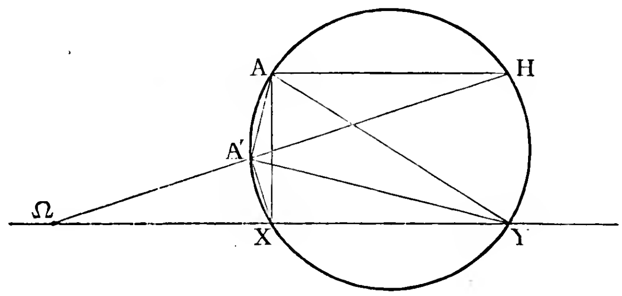

Fig. 18.

impulsive wrench of unit intensity at X, and let β be the corresponding quantity for y.

It may be easily shown that the triangle $AA'X$ is similar to $YA'\Omega$, and that the triangle $AA'Y$ is similar to $XA'\Omega$; whence we obtain,

$$\frac{A'X}{AX} = \frac{\Omega A'}{\Omega Y}; \quad \frac{A'Y}{AY} = \frac{\Omega A'}{\Omega X}.$$

The unit wrench on A decomposes into components on X and Y of respective intensities,

$$\frac{AY}{XY}, \quad \frac{AX}{XY},$$

These will generate twist velocities,

$$\alpha \frac{AY}{XY}, \quad \beta \frac{AX}{XY}.$$

Let ω be the resulting twist velocity on A', then the components on X and Y must be equal to the quantities just written; whence,

$$\omega \frac{A'Y}{XY} = \alpha \frac{AY}{XY},$$

$$\omega \frac{A'X}{XY} = \beta \frac{AX}{XY},$$

and we obtain

$$\alpha = \omega \frac{\Omega A'}{\Omega X}; \quad \beta = \omega \frac{\Omega A'}{\Omega Y};$$

or,

$$\alpha : \beta :: \Omega Y : \Omega X;$$

we thus see that Ω is a fixed point wherever A and A' may be.

It also follows that

$$\omega \,.\, \Omega A'$$

is constant; whence we have the following theorem :—

Draw through the impulsive screw A a ray AH parallel to the homographic axis, then the ray from H to a fixed point Ω on the homographic axis will cut the circle in the instantaneous screw A', and the acquired twist velocity will be inversely proportional to $\Omega A'$.

If the twist velocity to be acquired by A' from a unit impulsive wrench on A be assigned, then $\Omega A'$ is determined: there will be two screws A', and two corresponding impulsive screws, either of which will solve the problem. The diameter through Ω finds the two screws about which the body will acquire the greatest and the least velocity with a given intensity for the impulsive wrench.

§ 30. **Twist Velocities on the Principal Screws.**—The quantities α and β, which are the twist velocities acquired by unit impulsive wrenches on the principal screws, can be expressed geometrically as follows (fig. 17) :—

Let ω be the twist velocity acquired on A' by the wrench on A, then, by the last article,

$$AY\alpha = A'Y\omega,$$

$$AX\beta = A'X\omega;$$

whence

$$\alpha : \beta :: \frac{A'Y}{A'X} \cdot \frac{AY}{AX}.$$

This ratio is the anharmonic ratio of the four points X, Y, A, A', that is, of X, Y, O, O'; whence, finally,

$$\alpha : \beta :: \frac{O'Y}{O'X} : \frac{OY}{OX}.$$

§ 31. **Another Investigation for the Twist Velocity acquired by an Impulse.**

We have just seen that

$$\alpha\, AY = \omega A'Y,$$

$$\beta\, AX = \omega A'X;$$

whence

$$\alpha\beta\, AX\,.\,AY = \omega^2 A'X\,.\,A'Y.$$

Let fall AP, $A'P'$, HQ perpendiculars on the homographic axis (fig. 19). Then, by the properties of the circle,

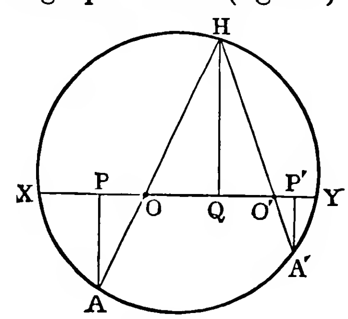

Fig. 19.

$$AX\,.\,AY : A'X\,.\,A'Y :: AP : A'P';$$

so that $\alpha\beta\, AP = \omega^2 A'P'.$

By similar triangles,

$$\alpha\beta\,.\,\frac{AO}{OH}\,.\,HQ = \omega^2\frac{O'A'}{O'H}\,.\,HQ;$$

whence,

$$\omega^2 = \alpha\beta\,\frac{OA\,.\,O'H}{O'A'.\,OH} \propto \frac{O'H^2}{OH^2};$$

or, as before (§ 28),

$$\omega \propto \frac{O'H}{OH}.$$

It will be noticed that, for this investigation, H may have been chosen *arbitrarily* on the circle. We thus see that, besides the two points O and O', there will be a whole system of pairs of points which may be employed for finding the instantaneous screw, and for determining the instantaneous twist velocity.

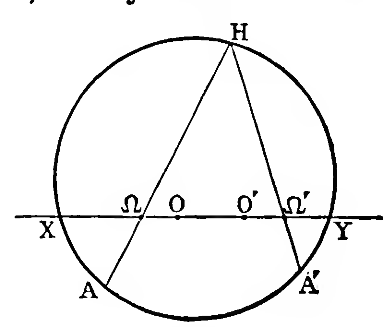

Fig. 20.

If we choose any two points (fig. 20), Ω and Ω', so that

$$(X\Omega\Omega'Y) = (XOO'Y);$$

then A being given, $A\Omega$ finds H, and $H\Omega'$ finds A', while the acquired twist velocity is proportional to $\Omega'H \div \Omega H$. We can suppose Ω to go to infinity,

and thus obtain the construction used in § 29. A similar construction is obtained when Ω' is at infinity.

The two points, A and A', will divide the arc cut off by XY in a constant anharmonic ratio, for the pencil $H\ (X\Omega\Omega' Y)$ always preserves the same anharmonic ratio as H moves round the circle.

§ 32. **A Curious Special Case.**—If η be an impulsive screw, and if α be the corresponding instantaneous screw, it will *not* usually happen that when α is the impulsive screw η is the corresponding instantaneous screw. If, however, in even a single case, it be true that the impulsive screw and the instantaneous screw are interchangeable, then the relation will be universally true.

Let Ω and Ω' (fig. 21) be a pair of points belonging to the system described in § 31. Then A being given, A' is found. If A' is similarly to find A, then the figure shows at once that Ω must lie on the polar of Ω', and, consequently, Ω and Ω' are conjugate points with respect to the circle; or, what comes to the same thing, they divide XY harmonically. The same must be true of each pair of points Ω and Ω', and therefore of O and O', and we have the following theorem :—

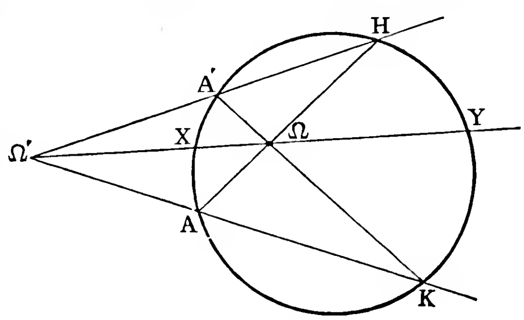

Fig. 21.

If the points O and O' be harmonic conjugates of the points where the homographic axis intersects the circle, then every pair of instantaneous and impulsive screws on the cylindroid are interchangeable.

We might, perhaps, speak of this condition of the system as one of *dynamical involution.* In this remarkable case an impulsive wrench of unit intensity applied to one of the principal screws of inertia will generate an equal and opposite velocity to that which would have been produced if the wrench had been applied to the other principal screw. The construction for the pairs of related screws becomes still more simplified by the theorem, that—

When the system is one of dynamical involution, the chord joining an impulsive screw with its instantaneous screw passes through the pole of the homographic axis.

We may take the opportunity of remarking, that dynamical involution is not confined to the system of the second order. It may be extended to a rigid body with any number of degrees of freedom, or even to any system of rigid bodies. Whenever it happens that the relation of impulsive screw and instantaneous screw is interchangeable in one case, it is interchangeable in every case.

For, let $\theta_1, \ldots \theta_n$ be the co-ordinates of an instantaneous screw, then (*Theory of Screws*, p. 60) the corresponding impulsive screw has for co-ordinates,

$$\frac{u_1^2}{p_1}\theta_1, \ldots \frac{u_n^2}{p_n}\theta_n;$$

and if this latter were regarded as an instantaneous screw, then its impulsive screw would be

$$\frac{u_1^4}{p_1^2}\theta_1, \ldots \frac{u_n^4}{p_n^2}\theta_n;$$

but as this is to be only

$$\theta_1, \ldots \theta_n,$$

we must have

$$\frac{u_1^4}{p_1^2} = \frac{u_2^4}{p_2^2} = \ldots \frac{u_n^4}{p_n^2},$$

which shows that if the theorem be true for one pair it is true for all. The conditions, of course are, that any one of the following systems of equations be satisfied :—

$$\pm\frac{u_n^2}{p_1} = \pm\frac{u_2^2}{p_2} = \ldots \pm\frac{u_n^2}{p_1}.$$

§ 33. **A more Simple Construction for the Twist Velocity acquired by an Impulse.**—Reverting to the general case, we find that the chord AA' (fig. 22) is cut by the homographic axis at T, so that the square of the acquired twist velocity is proportional to the ratio of TA to TA'.

For, with the construction in § 29, draw HQ parallel to AT; then,

$$HQ : A'T :: H\Omega : A'\Omega,$$

$$\frac{AT}{A'T} = \frac{H\Omega}{A'\Omega} \propto \frac{1}{A'\Omega^2};$$

but we showed, in the article referred to, that $A'\Omega$ varies inversely as the acquired twist velocity, whence the theorem is proved.

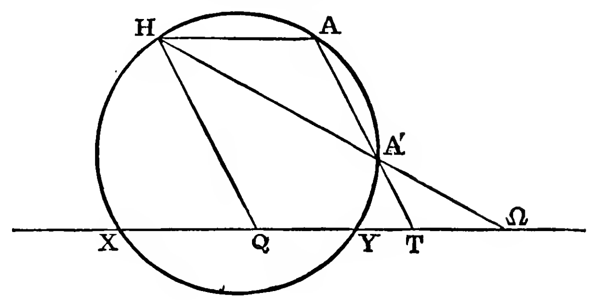

Fig. 22.

This is, in one respect, the simplest construction, for it only involves the chord AA' and the homographic axis.

The chord AA' must envelop a conic having double contact with the circle (fig. 23), for this is a general property of the chord uniting two corresponding points, A and A', of two homographic systems. Let I be the point of contact (fig. 23). Then AA' is divided harmonically in I and T; for, if XY be projected to infinity, the two conics become concentric circles, and the tangent to one meets it at the middle point of the chord in the other; the ratio is therefore harmonic, and must be so in every projection; whence,

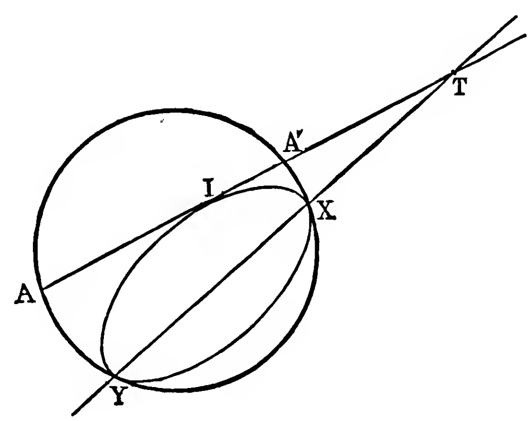

Fig. 23.

$$\frac{AI}{A'I} = \frac{AT}{A'T};$$

but the last varies as the square of the twist velocity acquired, and hence we see that—

The chord joining any impulsive screw A to the corresponding instantaneous screw A' envelops a conic, and the point of contact, I, divides the chord into segments, so that the ratio of AI to $A'I$ is proportional to the square of the twist velocity acquired about A' by the unit impulsive wrench on A.

§ 34. **Constrained Motion.**—We can now give another demonstration of the theorem in the *Theory of Screws*, p. 56, which is thus stated :—

If a body, *constrained* to twist about the screw α, be acted upon by an impulsive wrench on the screw η, then the twist velocity acquired varies as

$$\frac{\varpi_{\alpha\eta}}{u_\alpha^{\,2}}.$$

The numerator in this expression is the virtual coefficient of the two screws, and the denominator is the function of § 21, which is proportional to the kinetic energy of the body when twisting about α with the unit of twist velocity.

Let A' be α and I, η (fig. 24), and let A be the impulsive screw which would correspond to A' if the body were free to choose its natural instantaneous screw on the cylindroid defined by A and A'. Let K be reciprocal to A'.

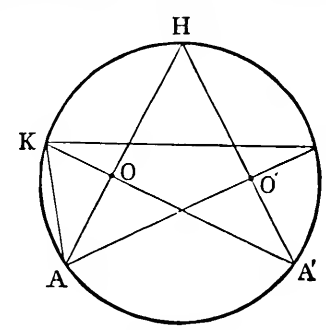

Fig. 24.

The impulsive wrench on I is decomposed into components on K and A. The former is neutralized by the constraints; the latter has the intensity

$$\frac{KI}{KA};$$

whence the twist velocity ω, acquired by A', is (§ 28) proportional to

$$\frac{KI}{KA}\cdot\frac{HO'}{HO};$$

but, by geometry,

$$KA = \frac{A'H}{OA'}\,.\,OA,$$

whence, §§ 13, 21,

$$\omega \propto \frac{KI\,.\,HO'OA'}{HO\,.\,HA'OA'},$$

$$\propto \frac{KI\,.\,OA'}{O'A'\,.\,HA'},$$

$$\propto \frac{\varpi_{\alpha\eta}}{u_\alpha^{\,2}}.$$

This is an interesting proof of a general theorem relating to any two screws.

§ 35. **Energy acquired by an Impulse.**—The kinetic energy acquired by a given impulse, using the same notation as before, is (*Theory of Screws*, p. 56),

$$\frac{\varpi^2_{\eta a}}{u_a^2}.$$

Let A be the impulsive screw, and A' the screw about which the body is *constrained* to twist. Draw the chord AOH (fig. 25), then, as A' varies,

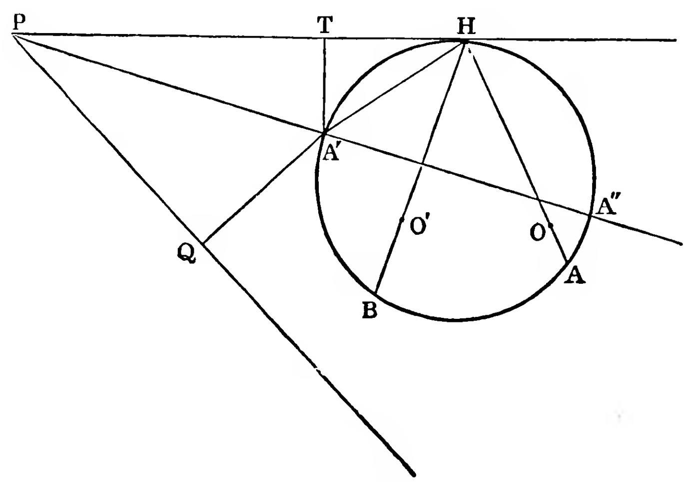

Fig. 25.

while A remains fixed, the virtual coefficient of A and A' varies as $A'H$. The square of this is proportional to $A'T$, the length of the perpendicular from A' on the tangent at H. If PQ be the axis of inertia, the value of u_a^2 is proportional to the perpendicular $A'Q$, and, accordingly, the kinetic energy acquired is proportional to

$$\frac{A'T}{A'Q}.$$

Any ray through P, the intersection of the axis of inertia with the tangent at H, cuts the circle in two points, A' and A'', either of which will accept the same kinetic energy from the given impulse.

§ 36. **Euler's Theorem.**—If the body be permitted to select the screw about which it will commence to twist, then Euler's theorem states that the body will commence to move with a greater kinetic energy than if it be restricted to some other screw. By drawing the tangent from P (not, however, shown in the figure) we obtain the point B, and at B it is obvious

that the ratio of the perpendiculars on PH and PQ is a maximum, and, consequently, the kinetic energy is greatest. It follows from Euler's theorem that B will be the instantaneous screw corresponding to D as the impulsive screw. The line BH is the polar of P, and, consequently, BH must contain O', the pole of the axis of inertia. We are thus again led to the well-known construction (§ 27) for the instantaneous screw B; that is, draw AOH, and then $HO'B$.

§ 37. **To determine a Screw that will acquire a given Twist Velocity under a given Impulse.**—The impulsive screw being given, and the intensity of the impulsive wrench being one unit, the acquired twist velocity (§ 34) will vary as (Fig. 25),

$$\frac{A'H}{A'Q}.$$

If, therefore, the twist velocity be given, this ratio is given. A' must then lie on a given ellipse, with H as the focus and the axis of inertia as the directrix. This ellipse will intersect the circle in *four* points, any one of which gives a screw which fulfils the condition proposed in the problem.

§ 38. **Principal Screws of the Potential.**—Let us suppose that a body having two degrees of freedom is in a position of stable equilibrium under the influence of a conservative system of forces. If the body be displaced by a small twist, it will no longer be in a position of equilibrium, and a wrench has commenced to act upon it. This wrench can always, by suitable composition with the reactions of the constraints, be replaced by an equivalent wrench on a screw of the cylindroid (see § 23).

For every point α, corresponding to a displacement screw, we have a related point, η, corresponding to the screw about which the wrench is evoked. This relation is of the one-to-one type, and it will now be proved that the system of screws α is homographic with the corresponding system η. The proof will follow, almost word for word, that already given in § 24, for impulsive and instantaneous screws.

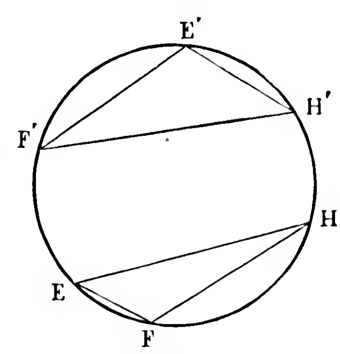

Fig. 26.

Let E be a displacement screw (fig. 26) on which a small unit of twist

evokes a wrench of intensity e on E'; let f be the similar quantity for another pair of screws, F and F'.

A twist of unit amplitude about H may be decomposed into components,

$$\frac{HF}{EF}, \frac{HE}{EF},$$

about E and F, respectively.

These will evoke wrenches on E' and F' of the intensities

$$e\frac{HF}{EF}, f\frac{HE}{EF},$$

respectively. But this pair of wrenches are to compound into a wrench of intensity h on H', and consequently we have

$$h\frac{H'F'}{E'F'} = e\frac{HF}{EF},$$

$$h\frac{H'E'}{E'F'} = f\frac{HE}{EF};$$

whence,

$$\frac{HF}{HE} : \frac{H'F'}{H'E'} :: f : e.$$

If we take another pair of points, K and K', we have

$$\frac{HF}{HE} : \frac{KF}{KE} :: \frac{H'F'}{H'E'} : \frac{K'F'}{K'E'};$$

whence,

$$(HKFE) = (H'K'F'E').$$

The anharmonic ratio of any four points in one system is equal to that of their correspondents, and the two systems are therefore homographic.

The homographic axis intersects the circle in two points, which are the principal screws of the potential, *i. e.* a twist about either evokes a wrench on the same. Of course it will be understood that this homographic axis is distinct from that considered in previous articles. The two axes intersect in O the pole of the axis of pitch.

§ 39. **Work done by a Twist.**—Suppose that the body, when in equilibrium under the system of forces, receives a twist of small amplitude α'

about any screw α, a quantity of work is expended, which we shall denote by

$$Fv_\alpha^2\alpha'^2.$$

In this, F is a constant, whose dimensions are a mass divided by the square of a time, and v_α is a linear magnitude specially appropriate to the screw α, and depending also upon the system of forces (*Theory of Screws*, p. 69). We may compare and contrast the three quantities, p_α, u_α, v_α: each is a linear magnitude specially correlated to the screw α. The first and simplest, p_α, is the pitch of the screw, and depends on the geometrical nature of the constraints; u_α involves also the mass of the body, and the distribution of the mass relatively to α; v_α, still more complicated, depends also on the system of forces.

§ 40. **Law of Distribution of v_α.**—As we follow the screw α around the circle, it becomes of interest to study the corresponding variations of the linear magnitude v_α. We have already found a very concise representation of p_α and u_α by the axis of pitch and the axis of inertia, respectively. We shall now obtain a precisely similar representation of v_α by the aid of the *axis of potential.*

It is shown (*Theory of Screws*, p. 69) that v_α^2 must be a quadratic function of the co-ordinates; we may therefore apply to this function the same reasoning as we applied to u_α^2 (§ 21). We learn that v_α^2 is at each point proportional to the perpendicular on a ray, which is the axis of potential.

Thus, if A (fig. 27) be the screw, the value of v_α^2 is proportional to AP, the perpendicular on PT; if O'' be the pole of the axis of potential, then, as in § 11, we can represent the value of v_α^2 by the product $AO''.AA'$.

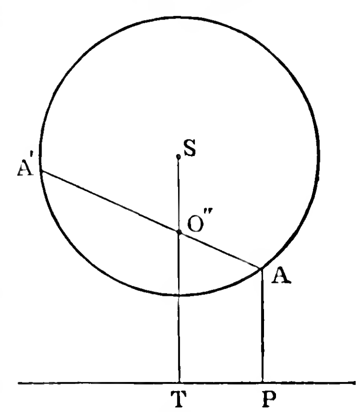

Fig. 27.

§ 41. **Conjugate Screws of Potential.**—In general the energy expended by a small twist from a position of equilibrium can be represented by a

quadratic function of the co-ordinates of the screw. If, moreover, the two screws of reference form what are called *conjugate screws of potential* (*Theory of Screws*, pp. 66, 73), then the energy is simply the sum of two square terms. The necessary and sufficient condition that the two screws shall be so related is, that their chord shall pass through O''.

Another property of two conjugate screws of potential is also analogous to that of two conjugate screws of inertia. If A and A' be two conjugate screws of potential, then the wrench evoked by a twist around A is reciprocal to A', and the wrench evoked by a twist around A' is reciprocal to A (*Theory of Screws*, p. 66).

§ 42. **Determination of Wrench Evoked by a Twist.**—The theorem just enunciated provides a simple means of discovering the wrench which would be evoked by a given small twist away from a position of equilibrium.

Let A (fig. 28) be the given screw; join AO'', and find H; then the required screw A' must be reciprocal to H, and is, accordingly, found by drawing the chord HA' through O.

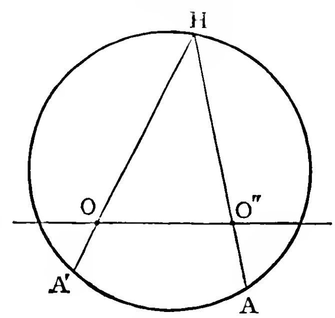

Fig. 28.

The axis OO'' is of course the homographic axis of § 38. We need not here repeat the demonstrations of § 28, which will apply, *mutatis mutandis*, to the present problem. We see that the ratio of the intensity of the wrench to the amplitude of the twist is proportional to

$$\frac{HO}{HO''}.$$

The other constructions of a like character can also be applied to this case.

§ 43. **Harmonic Screws.**—If after displacement the rigid body be released, and small oscillations result, the present geometrical method permits us to study the resulting movements.

It must first be shown that there are two special screws on the surface, each of which possesses the property of being an *harmonic screw*. If a body

be displaced from rest by a small twist about an harmonic screw, and if it also receive any small initial twist velocity about the same screw, then the body will continue for ever to perform harmonic twist oscillations about the same screw.

The two harmonic screws are X and Y, the intersections with the circle of the axis passing through the pole of the axis of inertia O', and the pole of the axis of potential O''. (Fig. 29.)

For, suppose the body receives a small initial displacement about X, this will evoke a wrench on H, found by drawing $XO''Y$ and YOH. But the effect of a wrench on H will be to produce twist velocity about a screw found by drawing HOY and $YO'X$, *i. e.* X itself. Hence the wrench evoked can only make the body still continue to twist about X, and harmonic vibration on X will be the result. Similar reasoning, of course, applies to Y.

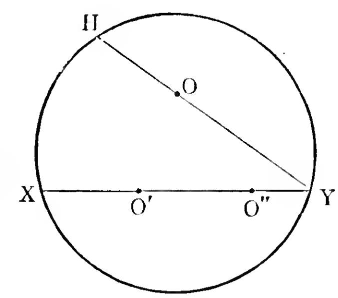

Fig. 29.

Small Oscillations in general.—Whatever be the initial displacement, or initial twist velocity of the body, they can be always decomposed into components on X and Y. The resulting small oscillations can thus be produced by compounding simple harmonic twist oscillations about X and Y.

If it should happen that O' and O'' become coincident, then every screw would be a harmonic screw.

If O and O' coincided, then every screw would be a principal screw of inertia.

If O and O'' coincided, then every screw would be a principal screw of potential.

§ 45. **Conclusion.**—The object proposed in this Memoir has now been completed. It has been demonstrated that the representative circle affords a concise method of exhibiting every problem in the Dynamics of a Rigid System with two degrees of freedom, so long as the body remains near its initial position. The geometrical interest of the inquiry is found mainly to depend on the completely *general* nature of the constraints. If the

constraints be specialized to those with which mechanical problems have made us familiar, it will frequently be found that the geometrical theory assumes some extreme and uninteresting type. For instance, a case often quoted as an illustration of two degrees of freedom, is that of a body free to rotate around an axis, and to slide along it. The representative circle has then an infinite radius, and the finite portion thereof is merely a ray perpendicular to the axis of pitch. The geometrical theory becomes attenuated, and entirely devoid of instruction.

THE END.

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